İSTANBUL BİLGİ ÜNİVERSİTESİ

Murat Sertel Center for Advanced Economic Studies

Working Papers

Year 2013 No 3

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MS Center Working Paper 2013 No3 manuscript No.
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Separable Voting Rules and the Strong Referendum Paradox

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The date of receipt and acceptance will be inserted by the editor

Abstract This paper provides a model of indirect elections where voters having weak orders as preferences over finitely many alternatives are distributed across a given set of districts. In each district preferences are aggregated into a district preference, and a voting rule selects one or several alternatives from the profile of district preferences. The referendum paradox holds at some profile and some distribution of voters across districts if the outcome of indirect elections does not coincide with the one of direct elections. We prove that whenever an indirect election procedure is separable, it is exposed to the referendum paradox if and only if it is exposed to a stronger version of the referendum paradox, where direct and indirect elections give different outcomes *for any distribution of the voters across districts*. We prove that many indirect elections based on a tournament solution are separable, whereas some based on a scoring rule are not. Finally, we show that all indirect elections based on a scoring rule are exposed to the strong referendum paradox

Key words Voting Paradox – Referendum Paradox – Representative Democracy - Gerrymandering - *JEL Class D 72*

1 Introduction

Representative democracy is based on indirect elections: voters are distributed across districts, each having one or several representatives, and final decisions are made from representatives' preferences, which aggregate in some way voters' preferences in each district. A well-known drawback of indirect elections is that they may give an outcome inconsistent with the one that would prevail in direct elections. "*Representative democracy is at best a working model of direct democracy and is most successful when it generates decisions as close as possible to those that would be generated in a direct democracy. [...] It is direct democracy (actual or ideal) that is used as a measuring rod*" (Chamberlin and Courant, 1983). Two main reasons explain why indirect elections may fail at being a working model of direct democracy. One is that the prevailing voting rules, such

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A first draft of the paper was written while the third author was visiting the Department of Economics at Bogazici University, whose support is gratefully acknowledged. Special thanks are addressed to Remzi Sanver for his valuable comments.

as the list-proportional rule or the single-transferable-vote rule, may distort the citizenry wills[1]. The second is that the way voters are distributed across districts may determine the outcome. This can be illustrated by the following example, known as the *referendum paradox* (Nurmi (1998, 1999), Laffond and Lainé (2000)). Suppose that a 15-voter electorate has to decide upon a 'yes' or 'no' decision, where 1 stands for the former and 0 for the latter. The following table describes the voters' preferences (each cell represents one individual, where 0 means that she prefers 0 to 1):

Table 1

D_1	D_2	D_3	D_4	D_5
1	1	0	0	0
0	1	1	0	0
1	0	1	0	0

The referendum outcome under simple majority rule is 0. Consider now a two-tiers election where voters are distributed across a given set of districts. In each district, the outcome is chosen according to the majority will, while the final outcome is the decision preferred in a majority of districts. In the case where the 15 voters are apportioned into the five districts $D_1, ..., D_5$, 1 is preferred to 0 in a majority of districts, and therefore 1 is elected. Hence, the indirect elections outcome contradicts the majority will in the whole electorate.

The referendum paradox is often associated with gerrymandering, defined as a deliberate design of districts to influence the outcome of two-step elections. Gerrymandering may operate by concentrating as many voters of one type into a single district to reduce their influence in other districts, or by spreading out voters of a particular type among many districts in order to prevent a sufficiently large voting block in any particular district. Gerrymandering may explain why the design of districts is always controversial. It is implicitly assumed in disputes that districts can be designed in order to avoid the referendum paradox. This is actually true in the case of a dichotomous choice. In the above example, if the 0-voter in D_1 swaps her location with a 1-voter in D_2 (equivalently, if these two voters' names are permuted), then direct and indirect outcomes become identical.

In this paper we argue that in many non-dichotomous choice situations, it may be impossible to avoid the referendum paradox even when the population in each district can be freely chosen. Indeed, direct and two-tiers elections can lead to different results *for any distribution of voters across a given set of districts.* We call *strong referendum paradox* any such situation.

Our analysis of the strong referendum paradox is based on a general model of indirect elections with variable numbers of alternatives and voters, which loosely speaking works as follows. Voters have preferences over finitely many alternatives represented by a weak order, and are distributed across a given set of districts. Voters' preferences in each districts are aggregated into a complete binary relation called district preferences, and final choices are computed from the profile of district preferences by means of a voting rule, which selects a subset of alternatives. Combining district-wise preference aggregation with a voting rule defines a *constitution.* Special attention is paid to two classes of constitutions. A constitution is based on a tournament solution if the voting rule is a tournament solution and if district preferences are defined as the majority tournament built from the voter preference profile in that district. A constitution is based on a scoring rule if the voting rule is a scoring rule and if district preferences are defined as the weak order built from voters' preferences by means of that scoring rule. Our model significantly departs from the ones suggested by Chambers (2008, 2009) and by Bervoets and Merlin (2012), where voters submit single votes rather than weak orders, and voting rules selects one alternative only.

Our main result (Theorem 1) is that the referendum paradox is essentially equivalent to the strong referendum paradox for any separable constitution where the voting rule is anonymous. Separability is a property for voting rules which can be illustrated by means of the following example: suppose that there are two issues, one being the level of consumption tax, the other being the number of vacation days in primary

[1] See Chamberlin and Courant (1983) and Feld and Grofman (1986).

schools. Suppose that much higher importance is given by each citizen to the tax issue, in the sense that preferences on overall programs (that is pairs of tax and vacation decisions) coincide with preferences over tax whenever tax decisions differ, and with preferences over vacation duration otherwise. Separability states that issue-wise choice leads to the same outcome as choice over entire programs. Moreover, separability is weaker than the composition-consistency property introduced in Laffond *et al.* (1996).

Theorem 1 is stated for any indirect election, and thus applies to institutions that involve several successive vote delegations or district-specific construction of district preferences. What matters only is the separability property. This motivates investigating the existence of separable constitutions in our model of two-step elections. We prove that many tournament solutions, as well as the Borda rule are separable. Moreover, we prove that every constitution based on a separable tournament solution is itself separable. Since every constitution based on a tournament solution is exposed to the referendum paradox, we deduce that every constitution based on a separable tournament solution is exposed to the strong referendum paradox. Finally, separability is not necessary for the strong referendum paradox to prevail. Indeed, we show that while many scoring rules are not separable, every constitution based on a scoring rule is exposed to the strong paradox.

The paper is organized as follows. Our model of indirect elections is described in Part 2. Our main result is presented in Part 3. The existence of separable tournament solutions and scoring rules is studied in Part 4. Part 5 is devoted to the analysis of the strong referendum paradox for constitutions based on scoring rules. We conclude the paper with several additional comments.

2 The strong referendum paradox

In its initial version, the referendum paradox relates to dichotomous collective choice based on simple majority voting. As such, it appears as a specific compound-majority paradox, which shows how a choice can be sensitive to a two-stage aggregation procedure (see Nurmi (1998,1999) for a discussion of the relationship between the referendum paradox and other compound-majority paradoxes, such as the Anscombe's paradox and the Ostrogorski paradox). When a society faces two possible choices, and when voters are distributed across districts, the outcome of simple majority voting in the population is defeated by a majority of voters in a majority of districts. This naturally calls for generalizing this idea to situations with more than two alternatives, and where choices no longer relates to the majority will. We describe below how the referendum paradox can be formalized to any voting rule defined for finitely many social alternatives (section 2.1), and we define within this framework the strong referendum paradox (section 2.2). We relate in section 2.3 our model to a different approach introduced by Chambers (2008, 2009).

2.1 Referendum paradox in a general model of representative democracy

Representative democracy is defined as a two-stage collective choice procedure. In the first stage, voters are distributed into districts, each being associated with one or several representatives. Representatives' preferences are built from district preferences, leading to a representative profile. Based on this representative profile, final decisions are made in the second stage. Hence, a model of representative democracy must describe how voters are distributed across districts, how representatives aggregate voter's preferences, and how final decisions are taken from representatives' preferences. What follows is a general model of representative democracy for variable numbers of voters and alternatives.

We start with the following notations and definitions.

We denote by $\mathbb{N}$ the set of natural numbers, and by $\varDelta$ the set of finite subsets of $\mathbb{N}$. We consider situations where a finite set of individuals $N = \{1, ..., i, ..., n\} \in \varDelta$ faces a finite choice set. A choice set is an element ω of $\varDelta$. The set of weak orders (resp. linear orders, complete binary relations) over choice set ω is denoted by $\mathcal{R}(\omega)$ (resp. $\mathcal{L}(\omega)$, $\mathcal{C}(\omega)$). Given a choice set ω together with a set of voters N, voter i has preferences over ω

represented by a weak order π_i. We write $x\ \pi_i\ y$ (resp. $x\ \pi_i^+\ y$, $x\ \pi_i^=\ y$) when i prefers (resp. strictly, equally) alternative x to alternative y in choice set ω. A preference profile on ω is a vector $\pi = (\pi_1, ..., \pi_n) \in \mathcal{R}(\omega)^N$. For any $N, \omega \in \Delta$, for any $\pi \in \mathcal{R}(\omega)^N$, for any $M \subset N$, we denote by π_M the restriction of π to M. Since the numbers of voters and of alternatives is variable, the set of all possible profiles over ω is $\cup_{N\in\Delta}\mathcal{R}(\omega)^N$ and the set of all possible profiles is $\cup_{N,\omega\in\Delta}\mathcal{R}(\omega)^N$. A function F from $\cup_{N,\omega\in\Delta}\mathcal{R}(\omega)^N$ to $\cup_{\omega\in\Delta}(2^\omega - \varnothing)$ is a voting rule if such that for any $\omega \in \Delta$ and any $\pi \in \cup_{N\in\Delta}\mathcal{R}(\omega)^N$, we have $\varphi(\pi) \in (2^\omega - \varnothing)$. Hence, given a choice set ω together with a set N of n voters, a voting rule maps each possible profile of n weak orders over ω to a non-empty subset of ω, called the winning set for π.

A voting rule F is *anonymous* if its winning set is not sensitive to the voters' names: for any permutation σ of N and for any profile π, one has $F(\pi) = F(\pi^\sigma)$, where π^σ is defined by $\pi^\sigma = (\pi_{\sigma(1)}, ..., \pi_{\sigma(i)}, ..., \pi_{\sigma(n)})$. Furthermore, F is *neutral* if winning sets are non-sensitive to the labelling of alternatives: for any $\omega, \omega' \in \Delta$ with $|\omega| = |\omega'|$, for any profile π over ω, for any bijection $f : \omega \to \omega'$, one has $F(\pi^f) = f(F(\pi))$, where π^f is the profile on ω' defined by $\forall i \in I$, $\forall x, y \in \omega$, $x\ \pi_i\ y \Leftrightarrow f(x)\ \pi_i^f\ f(y)$.

The set of partitions of $N \in \Delta$ into non-empty subsets is denoted by $\mathcal{P}(N)$. Elements of $D \in \mathcal{P}(N)$ are called districts. Pick a choice set ω, a set of voters N and a partition $D = \{D_1, ..., D_T\}$ of N into T districts. For sake of simplicity, we assume that each district has only one representative.[2] We define representatives' preferences in a very general way, and give below some illustrative examples. Given a profile π over ω, preferences of representative t of D_t are described by a complete binary relation over ω obtained from π_{D_t} by means of some function δ_t from π_{D_t} to $\mathcal{C}(\omega)$. We assume that when D_t involves only one voter with preference π_i, then π_i is also district representative preference. Gathering all representatives' preferences leads to the representative profile $\delta(\pi, D) = (\delta_1(\pi_{D_1}), ..., \delta_T(\pi_{D_T})) \in \mathcal{C}(\omega)^T$. This motivates the following definition:

Definition 1 *A representation function is a function* $\delta : \cup_{N,\omega\in\Delta}\mathcal{R}(\omega)^N \times \cup_{N\in\Delta}\mathcal{P}(N) \to \cup_{N,\omega\in\Delta}\mathcal{C}(\omega)^N$ *such that* $\forall N, \omega \in \Delta$, $\forall T \in \mathbb{N}$, $\forall D = \{D_1, ..., D_T\} \in \mathcal{P}(N)$, $\forall \pi \in \mathcal{R}(\omega)^N$,
(1) $\delta(\pi, D) \in \mathcal{C}(\omega)^T$
(2) $\forall t \in \{1, ..., T\}$, *if* $D_t = \{\pi_i\}$, *then* $\delta_t(\pi_{D_t}) = \pi_i$

Final decisions are taken by means of a voting rule which only depends on representatives' preferences. This leads to the following definition of a *constitution*, defined as a voting rule which conditions the final outcome to a partition into districts combined with a representation function.

Definition 2 *Given a representation function* δ*, a constitution is a voting rule* F_δ *is a constitution if* $\forall N \in \Delta$, $\forall D \in \mathcal{P}(N)$, $\forall \omega \in \Delta$, $\forall \pi, \pi' \in \mathcal{R}(\omega)^N$, $\delta(\pi, D) = \delta(\pi', D)$ *implies* $F_\delta(\pi) = F_\delta(\pi')$.

Pick any representation function δ together with a voter set N, a partition D of N into districts and a profile π. Since $F_\delta(\pi)$ obviously depends on D, we make a slight notation abuse by writing $F_\delta(\pi) = F_{D,\delta}(\pi)$. Moreover, if D^1 stands for the partition of N into singletons, it follows from definition of δ that $F_{D^1,\delta}(\pi) = F(\pi)$, which describes the outcome in direct democracy.

We are now ready to formalize the referendum paradox:

Definition 3 *A constitution* F_δ *is exposed to the referendum paradox if there exist* $N, \omega \in \Delta$*, a partition* D *of* N *into districts and a profile* $\pi \in \mathcal{R}(\omega)^N$ *such that* $F(\pi) \neq F_{D,\delta}(\pi)$.

The referendum paradox holds for some voting rule when district-wise preference profiles are aggregated into district preferences in such a way that the final outcome, which depends on district preferences only, differs from the one obtained when there is no vote delegation (each voter is herself a district).

We illustrate below our general model by means of two specific classes of constitutions.

[2] Allowing for sets of district representatives is possible at the cost additional notations. Since our results do not rest neither on the number of representatives nor their distribution across districts, we restrict our model to the simplest case.

2.1.1 Constitutions based on tournament solutions A tournament over a finite set ω is a complete and anti-symmetric binary relation γ over ω. If a and b are two elements of ω, we say that a defeats b if $(a, b) \in \gamma$. An alternative a is a (necessarily unique) *tournament winner* of tournament γ if a defeats all other alternatives in ω. Denoting by $\varGamma(\omega)$ the set of tournaments on ω, a tournament solution is a function $S : \cup_{\omega \in \varDelta} \varGamma(\omega) \to \cup_{\omega \in \varDelta}(2^{\omega} - \varnothing)$ such that for all $\omega \in \varDelta$ and all $\gamma \in \varGamma(\omega)$, we have $S(\gamma) \in (2^{\omega} - \varnothing)$ and $S(\gamma) = \{a\}$ whenever a is a tournament winner of γ. Hence, a tournament solution maps every tournament over some finite set to a non-empty subset of that set, and this subset reduces to the Condorcet winner whenever it exists.

Given an odd set N of voters together with a profile $\pi \in \mathcal{L}(\omega)^N$, we can associate to π the majority tournament $\gamma(\pi)$ on ω obtained by computing all pairwise majority comparisons of alternatives according to the simple majority rule. Formally, $\gamma(\pi)$ is defined on $\omega \times \omega$ by: for any $a, b \in \omega$, $(a, b) \in \gamma(\pi)$ iff $|\{i \in N : a\pi_i b\}| > \frac{|N|}{2}$. A voting rule F is *Condorcet* if $F(\pi) = \{a\}$ for all π where a is the tournament winner of $\gamma(\pi)$, called *Condorcet winner*. Moreover, F is a *tournament solution* if F is Condorcet and $\forall N, \omega \in \varDelta$, $\forall \pi, \pi' \in \mathcal{L}(\omega)^N$, $\gamma(\pi) = \gamma(\pi') \Rightarrow F(\pi) = F(\pi')$. A constitution F_δ is based on a tournament solution F if for all $N, \omega \in \varDelta$, for all $D = \{D_1, ..., D_T\} \in \mathcal{P}(N)$, and for all profiles $\pi \in \mathcal{R}(\omega)^N$ such that $\gamma(\pi)$ and all $\gamma(\pi_{D_t})$ are well-defined, we have $\delta_t(\pi, D_t) = \gamma(\pi_{D_t})$ for all $t \in \{1, ..., T\}$, $\delta(\pi, D) = \gamma(\gamma(\pi_{D_1}), ..., \gamma(\pi_{D_T}))$ and $F_{D,\delta}(\pi) = F(\delta(\pi, D))$.[3] In words, linear orders in each district are aggregated into the district majority tournament. Moreover, district majority tournaments are themselves aggregated into a majority tournament, where an alternative a defeats another alternative b if it does so in a majority of districts. This means that representatives report at the upper level all pairwise majority comparisons of alternatives prevailing in their district. Clearly, this type of constitution generalizes example given in Table 1 where the choice set ω is dichotomous. Laffond and Lainé (2000) show that every constitution based on a tournament solution is exposed to the referendum paradox, even when all districts have the same size and when all majority margins within the initial voters' profile are arbitrarily close to 75% from below.[4] However, if all majority margins are at least 75%, every constitution based on a tournament solution is immune to the paradox.

A variant of this model, leading to *sequential constitutions*, assumes that each district selects a set of winning alternatives, and that the set of all alternatives selected in at least one district forms the choice set in the second stage. Assuming that all district choices rest upon the same tournament solution F, we get $\delta(\pi, D) = \gamma(\pi, D) / \cup_{1 \leq t \leq T} F(\pi_{D_t})$ and $F_{D,\delta}(\pi) = F[\gamma(\gamma(\pi_{D_1}), ..., \gamma(\pi_{D_T})) / \cup_{1 \leq t \leq T} F(\pi_{D_t})]$. Laffond and Lainé (2000) get for this variant the same results as above. It is easy to prove that all sequential constitutions are exposed to the paradox. Indeed, consider profile π given in Table 2 below:

Table 2

1	1	2	3	4	0	10	9	8	7	6	5
2	1	5	6	7	0	10	9	8	4	3	2
3	3	5	8	9	0	10	7	6	4	2	1
4	2	6	8	10	0	9	7	5	4	3	1
5	4	7	9	10	0	8	6	5	3	2	1

We have 5 voters $\{1, ..., 5\}$ and 11 alternatives. Voter i's linear orders (in decreasing order) is given in row i. Suppose that all choices (district-wise and from representative profile) are made by means of the tournament solution F. It is easily checked that c_0 is a Condorcet winner of $\gamma(\pi)$. Therefore c_0 is the unique

[3] Since we allow voters to be indifferent and the number of voters may be even, the majority tournament may not be well-defined at some preference profile. Ignoring this possibility in the definition of a constitution based on majority tournament is inocuous as long as the main result of the paper is concerned. Hence, considering the case of weak tournaments, where ties are possible, is left as an exercise, which can be done along the lines drawn in Peris and Subiza (1999).

[4] The result in Laffond and Lainé (2000) is actually stronger, since it shows that Condorect constitutions and direct voting may lead to mutually disjoint outcomes.

winner in direct elections. Let $D = \{D_1, D_2, D_3\}$ with $D_1 = \{1, 2, 3\}$, $D_2 = \{4\}$ and $D_3 = \{5\}$. We get that $F(\pi_{D_1}) = \{1\}$, $F(\pi_{D_2}) = \{2\}$ and $F(\pi_{D_3}) = \{4\}$. Since $0 \notin \cup_{1\leq t\leq 3} F(\pi_{D_t})$, the referendum paradox holds.

2.1.2 Constitutions based on scoring rules Given a weak order π_i over ω together with $x \in \omega$, the indifference class of x is $I(x, \pi_i) = \{y \in \omega : x\pi_i^{=} y\}$. Then ω admits the partition $\mathcal{I}(\pi_i) = \{I(x, \pi_i), x \in \omega\}$ into indifference classes. Moreover, π_i induces the linear order P_i over $\mathcal{I}(\pi_i)$ defined by: $\forall x, y \in \omega$, $I(x, \pi_i)\ P_i\ I(y, \pi_i)$ if and only if $x\pi_i^{+} y$. We define the rank of $x \in \omega$ for π_i as $rank(x, \pi_i) = |\{I(y, \pi_i) \in \mathcal{I}(\pi_i) : I(y, \pi_i)\ P_i\ I(x, \pi_i)\}| + 1$. Consider profile $\pi \in \mathcal{R}(\omega)^N$, and define $I(\pi) = Max\{|\mathcal{I}(\pi_i)|, i \in N\}$. A score vector for π is an element $s = (s_1, ..., s_{|\mathcal{I}(\pi)|})$ of $\mathbb{R}_+^{|\mathcal{I}(\pi)|}$ such that $s_1 \geq s_2 \geq ... \geq s_{I(\pi)} = 0$ and $s_1 > s_{I(\pi)}$.

A voting rule F is a *scoring rule* if, for all $N, \omega \in \Delta$, for all profiles π on ω, there exists a score vector s for π such that $F(\pi) = ArgMax_{x\in\omega} S^\pi(x)$, where $S^\pi(x) = \sum_{i=1}^{n} s_{rank_{\pi_i}}(x)$.

A scoring rule chooses all those alternatives with maximal total score, a score s_k being given to an alternative x by individual i whenever x is given rank k in $i's$ preference. The Borda rule is defined by $s_k^\pi = I(\pi) - k$ for all $k = 1, ..., I(\pi)$.

Given a profile π over ω, given a scoring rule F with score vector s for $I(\pi)$, we define $\widetilde{F}(\pi)$ as the weak order over ω defined by: for all $x, y \in \omega$, $x\ \widetilde{F}(\pi)\ y$ if and only if $S^\pi(x) \geq S^\pi(y)$. Thus, $\widetilde{F}(\pi)$ ranks alternatives according to their decreasing total score in profile π.

A constitution F_δ is *based on a scoring rule* F if for all $N, \omega \in \Delta$, for all $D = \{D_1, ..., D_T\} \in \mathcal{P}(N)$, and for all profiles $\pi \in \mathcal{R}(\omega)^N$, we have $\delta(\pi, D) = (\widetilde{F}(\pi_{D_1}), ..., \widetilde{F}(\pi_{D_T}))$ and $F_{D,\delta}(\pi) = F(\delta(\pi, D))$. In words, voters' preferences in each district are aggregated into a district weak order built by means of the scoring rule F, and final decisions are the best alternatives according to F when applied to the profile of district weak orders.

As for constitutions based on a tournament solution, a sequential variant of constitution based on a scoring rule imposes final choices to be ranked first in at least one district. Formally, we have: $\forall N, \omega \in \Delta$, $\forall D = \{D_1, ..., D_T\} \in \mathcal{P}(N)$, $\forall \pi \in \mathcal{R}(\omega)^N$, $F_{D,\delta}(\pi) = F((\widetilde{F}(\pi_{D_1}), ..., \widetilde{F}(\pi_{D_T}))/ \cup_{1\leq t\leq T} F((\pi_{D_t}))$.

For an illustration, consider a constitution based on the Borda rule. Let $\omega = N = \{1, 2, 3, 4\}$ and voters' preference profile π be defined in Table 3 below:

Table 3

1	1	2	3	4
2	2, 4	3	1	
3	2	4	1	3
4	3	1	4	2

where two indifferent alternatives are put in the same cell. We have $I(\pi) = 4$, and therefore the score vector is $s = (3, 2, 1, 0)$. The reader will check that $F(\pi) = \{2\}$. If $D = \{D_1, D_2\} = \{\{1, 4\}, \{2, 3\}\}$, then we get from $\widetilde{F}$ the following profile $\delta(\pi, D)$ of district preferences:

Table 4

D_1	1	3	2	4
D_2	2	4	1, 3	

Hence $F(\delta(\pi, D)) = \{1, 2\} \neq F(\pi)$ and the referendum paradox holds.

We prove below that any constitution based on a scoring rule is exposed to a stronger version of the referendum paradox.

2.2 Strong referendum paradox: Definition

Consider a constitution exposed to the referendum paradox when the choice set is dichotomous. It is obvious to show that there always exists a way to redistribute voters across existing districts which gives the same outcome as in direct elections. Our purpose is to study whether this is still the case under non-dichotomous choice.

We say that a constitution is exposed to the strong referendum paradox if there is a choice set, a set of voters, a partition of the latter into districts and a preference profile such that the outcomes of direct election differs from the one of indirect election for any distribution of voters across districts. Formally, reshuffling voters across a fixed set of districts is equivalent to permuting voter's names. This motivates the following definition:

Definition 4 *A constitution F_δ is exposed to the strong referendum paradox if there exist $N, \omega \in \Delta$, a partition D of N into districts and a profile $\pi \in \mathcal{R}(\omega)^N$ such that $F(\pi) \neq F_{D,\delta}(\pi^\sigma)$ for all permutations σ of N.*

It is easy to show that every sequential Condorcet constitution is exposed to the strong paradox, provided that no restriction is put on district sizes. Indeed, consider again the 5-voter situation given in Table 2. For any distribution of voters across one district of size 3 and two districts of size 1, or equivalently for any permutation σ of voters' names, we have that $0 \notin \cup_{1 \leq t \leq 3} F(\pi^\sigma_{D_t})$. Since 0 is the winner of direct elections, the strong paradox holds.

2.3 Related literature

To the best of our knowledge, the strong referendum paradox has not been studied, while the only analysis of the referendum paradox based on our model above deals with Condorcet constitutions (Laffond and Lainé, 2000). Chambers (2008,2009) suggests a different approach where voters in variable number face a fixed choice set ω, and where constitutions use single-alternative ballots and where the representative function fulfills a "winners take all" assumption. Formally, each voter i submits a vote x_i defined as an alternative in ω. We add the following notation: For any $N \in \Delta$, for any $x \in \omega$, x^N stands for the vector in ω^N such that for all $i \in N$, $x_i = x$. A vote profile is a vector X in ω^N, and a choice function is defined as a function $\Psi : \cup_{N \in \mathcal{N}} \omega^N \rightarrow \omega$. Hence, a choice function selects a unique alternative from every vote profile on ω. Given $N \in \Delta$ together with a partition $D = \{D_1, ..., D_T\}$ in $\mathcal{P}(N)$ and a vote profile X, $\Psi(X_{D_t})$ defines the district vote in D_t, where X_{D_t} is the restriction of X to D_t. Each district has a single representative, who votes in the second stage as if all voters in her district had voted for the district vote. Moreover, it is assumed that each representative has at the upper level a weight equal to the size of her district. Finally, representative votes are aggregated into a final vote by means of Ψ. Within this setting, and adopting our terminology, a voting rule is exposed to the referendum paradox if there exists $N \in \Delta$, $D = \{D_1, ..., D_T\} \in \mathcal{P}(N)$ and $X \in \omega^N$ such that $\Psi(X) \neq \Psi(\Psi(X_{D_1})^{D_1}, ..., \Psi(X_{D_T})^{D_T})$. Chambers (2008) proves that unanimity, anonymity and immunity to the referendum paradox together characterize the very narrow class of partial priority rules. Loosely speaking, a partial priority rule is a type of unanimity rule, whereby choices are partially ordered. Note that the same characterization holds when partitions D are restricted those with equally-sized districts.

Bervoets and Merlin (2012) study within a similar framework of single ballot constitutions the property of gerrymander-proofness which closely relates to the referendum paradox. Using the same approach as the one above, suppose than instead of adopting the 'winners take all' hypothesis, district votes are computed from district vote profiles by means of some (maybe district-specific) voting rule, while the final vote is computed from district votes by using another voting rule. Given a fixed set of voters N, a partition $D = \{D_1, ..., D_T\}$ of N, and a fixed choice set ω, a constitution is defined by $(\Psi, \Psi_1, ..., \Psi_T)$ where Ψ_t is the voting rule in district D_t and Ψ is the voting rule applied to district vote profiles. Starting from a vote profile X in ω^N, the final outcome of a constitution is thus $\Psi(\Psi_1(X_{D_1}), ..., \Psi_T(X_{D_T}))$. A constitution is called gerrymander-proof if for all partitions D, and for any two permutations σ and σ' of N, we have $\Psi(\Psi_1(X^\sigma_{D_1}), ..., \Psi_T(X^\sigma_{D_T})) = \Psi(\Psi_1(X^{\sigma'}_{D_1}), ..., \Psi_T(X^{\sigma'}_{D_T}))$. Gerrymander-proofness holds when the outcome is the same whatever the distribution of voters across a given set of districts. Bervoets and Merlin (20012) show that, unless the constitution is constant (i.e. gives the same outcome regardless the vote profile), gerrymander-proofness combined with mild additional assumptions implies that each voter is pivotal, in the sense that she can change the winner by changing her vote at some unanimous profile.

Gerrymander-proofness is defined as follows within our setting:

Definition 5 *A constitution F_δ is gerrymander-proof if for all $N, \omega \in \Delta$, for all partitions D of N into districts, for all profiles $\pi \in \mathcal{R}(\omega)^N$ and for any two permutations σ and σ' of N, we have $F_{D,\delta}(\pi^\sigma) = F_{D,\delta}(\pi^{\sigma'})$.*

Clearly, if F is anonymous, then any constitution that is not gerrymander-proof is exposed to the referendum paradox.

3 Main Result

We present in this Part the main result of the paper, which links the separability property to the strong referendum paradox.

3.1 Separable voting rules: Definition

We first define the notion of separable voting rule. Separability focuses on the case where the choice set is a Cartesian product of finite sets, i.e. $\omega = \mathbb{X}_{j=1}^J \omega_j$, where $\omega_j \in \Delta$, $j = 1, ..., J$. Each set ω_j can be interpreted as a set of decisions regarding to a specific policy issue j, and there are J different policy issues.We first introduce *lexicographic preferences* over such choice sets. Given a linear order $\succ$ over $\{1, ..., j, ..., J)$ together with alternative $x = (x_1, ..., x_J)$ in $\omega = \mathbb{X}_{j=1}^J \omega_j$, we define $x^\succ = (x_1^\succ, ..., x_J^\succ)$ as the alternative obtained from x by ranking coordinates according to $\succ$.

Definition 6 *A preference profile $\pi = (\pi_1, ...\pi_n)$ is lexicographic over $\omega = \mathbb{X}_{j=1}^J \omega_j$ if there exists a linear order $\succ$ on $\{1, ..., J\}$ and J profiles $\pi^j = (\pi_1^j, ...\pi_n^j)$ over ω_j, where $j = 1, ..., J$, such that for all individuals i and for any two alternatives $x = (x_1, ..., x_J)$ and $y = (y_1, ..., y_J)$, one has $x\ \pi_i^+\ y$ if and only if $\exists j \in \{1, ..., J\}$ such that $x_j^\succ\ \pi_i^{j+}\ y_j^\succ$ and $x_k^\succ\ \pi_i^{k=}\ y_k^\succ$, for $k = 1, ..., j-1$.*

If π is lexicographic on $\omega = \mathbb{X}_{j=1}^J \omega_j$, where $1 \succ 2 \succ ... \succ J$ then for any $1 < k \leq J$, each preference π_i induces a weak order $\pi_i \mid \mathbb{X}_{j=k}^J \omega_j$ over $\mathbb{X}_{j=k}^J \omega_j$ defined by: $\forall y, z \in \mathbb{X}_{j=k}^J \omega_j$, $[y\ \pi_i \mid \mathbb{X}_{j=k}^J \omega_j\ z] \Leftrightarrow (x, y)\ \pi_i\ (x, z)$ for at least one x in $\mathbb{X}_{j=1}^{k-1} \omega_j$.

In a lexicographic preference profile over a Cartesian product of choice sets, two vectors x and y are compared according to the following rule: there exists a 'priority' order $\succ$ over coordinates (issues) common to all voters, so that x and y can be rewritten as $x^\succ$ and $y^\succ$, by ranking their coordinates in decreasing priority order ; then x is preferred to y if at the first coordinate j where $x^\succ$ and $y^\succ$ are not equally preferred, $x_j^\succ$ is better than $y_j^\succ$ according to the preference π^j defined over $\omega_j^\succ$.

Separability states that winning sets at lexicographic profiles are Cartesian product of issue-wise winning sets.

Definition 7 *A voting rule F is separable if, for any choice set $\omega = \mathbb{X}_{j=1}^J \omega_j$ and for any lexicographic profile π on ω, $F(\pi) = \mathbb{X}_{j=1}^J F(\pi^j)$.*

Separability describes a natural way to choose in the very specific case of lexicographic preferences: if alternatives are vectors of issue-wise decisions, and if all individuals agree on some priority order over issues and compare alternatives lexicographically according to this priority order, the choice procedure can be organized as a sequence of issue-wise choice procedures. Note that separability imposes nothing when profiles are not lexicographic.

3.2 Strong referendum paradox: Separable and anonymous voting rules

We start with a general result on separable voting rules. Proposition 1 below shows that if two separable choice rules have different (resp. disjoint) outcomes at a preference profile defined on some choice set, then there exists another profile on another choice set for which the same situation prevails *for all possible permutations of the voters' names.*

Proposition 1 *Let F and G be two separable voting rules such that there exists a choice set ω together with a profile π on ω such that $F(\pi) \neq G(\pi)$ (resp., $F(\pi) \cap G(\pi) = \emptyset$) . Then, there exist another choice set ω^* and a profile π^* on ω^* such that $F(\pi^{*\sigma}) \neq G(\pi^{*\sigma})$ (resp. $F(\pi^{*\sigma}) \cap G(\pi^{*\sigma}) = \emptyset$) for any permutation σ of I.*

Proof Let $\omega^* = \mathbb{X}_{1 \leq t \leq T}\omega_t$, where $\omega_t = \omega$ for all t, and where T is the number of permutations $\sigma_1, ..., \sigma_T$ of I. Let $\pi^* = (\pi_1^*, ..., \pi_n^*)$ be the lexicographic profile on ω^* with $1 \succ 2 \succ ... \succ T$ and such that for all individuals i and all $t \in \{1, ..., T\}$, $\pi_i^{*t} = \pi_i^{\sigma_t}$, where $\pi^{*t} = (\pi_1^{*t}, ..., \pi_n^{*t})$ is the associated profile on ω_t. Let σ be any permutation of I, and let σ_{Id} denote the identity mapping from I to I. Then, there exists $t(\sigma) \in \{1, ..., T\}$ such that $\sigma \circ \sigma_{t(\sigma)} = \sigma_{Id}$. It follows from $F(\pi) \neq G(\pi)$ together with $\omega_{t(\sigma)} = \omega$ and $\sigma \circ \sigma_{t(\sigma)} = \sigma_{Id}$, that $F((\pi^{*t(\sigma)})^\sigma) = F(\pi) \neq G((\pi^{*t(\sigma)})^\sigma) = G(\pi)$. Since F and G are separable, one has $F(\pi^{*\sigma}) = \mathbb{X}_{1 \leqslant t \leqslant T}F((\pi^{*t})^\sigma) \neq \mathbb{X}_{1 \leqslant t \leqslant T}G((\pi^{*t})^\sigma) = G(\pi^{*\sigma})$. The same argument applies to the case where $F(\pi) \cap G(\pi) = \emptyset$ □

As an immediate consequence of Proposition 1, we get

Theorem 1 *Let F_δ be an separable constitution such that F is anonymous. Then F_δ is exposed to the referendum paradox if and only if F_δ is exposed to the strong referendum paradox.*

Proof Let π be a profile on ω, let N and let $D \in \mathcal{P}(N)$ be such that $F(\pi) \neq F_{D,\delta}(\pi)$. Since F_δ is separable, it follows from Proposition 1 that there exists another choice set $\omega^* = \mathbb{X}_{1 \leq j \leq J}\omega_j$,where $\omega_j = \omega$ for all j together with a lexicographic profile π^* over ω^* such that $F(\pi^{*\sigma}) \neq F_{D,\delta}(\pi^{*\sigma})$ for any permutation σ of I. Since F is anonymous, then $F(\pi^{*\sigma}) = F(\pi^*)$ for all σ. Hence, $F(\pi^*) \neq F_{D,\delta}(\pi^{*\sigma})$ for all σ □

Since every constitution based on a tournament solution is exposed to the referendum paradox, and since a tournament solution is anonymous, Theorem 1 implies that every separable constitution based on a tournament solution is exposed to the strong paradox. We show below that many well-known tournament solutions are separable.

Theorem 1 can be interpreted in a much broader way. Our model of representative democracy involves only one delegation of votes. Moreover, in sequential constitutions district choices and final choices result from the same voting rule. We may consider more complex institutions with multi-level representation, where representatives at the first level are themselves distributed across several groups, and so on until final choices are made from the highest-level representative profile. We may also allow for level-specific voting rules, which successively select those alternatives to be considered at the next level. For all these constitutions the same result holds: provided separability and anonymity hold together, referendum and strong referendum paradoxes are equivalent. [5].

Another interpretation of Theorem 1 is the following. Suppose that direct elections are based on an anonymous and separable voting rule F. If G_δ is any separable constitution that does not agree with F given some distribution of voters across districts, then there is another voting situation where G_δ will not agree with F whatever the distribution of voters across the same set of districts.

[5] Note that anonymity is not strictly speaking needed. Indeed, the following result holds: If constitution F_δ is exposed to the referendum paradox at some profile π and F is anonymous at π, then F_δ is exposed to the strong referendum paradox.

Furthermore, since a constitution that is not gerrymander-proof is exposed to the referendum paradox, Theorem 1 implies that if G_δ is non-gerrymander-proof and separable, then G_δ will disagree with F whatever the distribution of voters. Violation of gerrymander-proofness pertains to a lack of anonymity: reshuffling voters across districts may change the results. Hence, Theorem 1 states that whenever a voting rule F is separable and anonymous, it is impossible to mimic F by means of a non-anonymous and separable voting rule G (interpreted as a constitution), where "mimic" means that F and G agree on every profile over any finite choice set, *even when one can freely permute the voters' names.*

This motivates the following conjecture. Since a constitution G_δ is generally not anonymous (i.e. not gerrymander-proof), we may still mimic F by considering the anonymous extension of G_δ. It associates with each pair {profile, choice set} the set of all alternatives chosen for at least one permutation of individuals. What Proposition 2 below states is that, as long as F and G_δ are separable, this procedure does not work: the F-winning set and the G_δ-winning set will not coincide at some profile. Hence, we cannot escape from the strong paradox by considering all distributions of voters across districts and computing the set of corresponding outcomes.

Proposition 2 *Let F and G be two separable voting rules such that $F(\pi) \neq G(\pi)$ for some profile π over a choice set ω. Then, if F is anonymous, there exist a choice set ω^* and a profile π^* over ω^* such that $F(\pi^*)$ $\neq \cup_{\sigma\in\Sigma} G(\pi^{*\sigma})$, where Σ denotes the set of all permutations of the individual set I.*

Proof Define ω^* and π^* as in proof of Proposition 1. Since G is separable, we can write $\cup_{\sigma\in\Sigma} G(\pi^{*\sigma}) = \cup_{\sigma\in\Sigma} \mathbb{X}_{1\leqslant t\leqslant T} G((\pi^{*t})^{\sigma}) = \cup_{\sigma\in\Sigma} \mathbb{X}_{1\leqslant t\leqslant T} G(\pi^{\sigma\circ\sigma_t})$. Moreover, since F is anonymous and separable, $F(\pi^*) = \mathbb{X}_{1\leqslant t\leqslant T} F((\pi^{*t})^{\sigma_t}) = \mathbb{X}_{1\leqslant t\leqslant T} F(\pi)$. Suppose $F(\pi) \nsubseteq G(\pi)$. Let $x \in F(\pi) - G(\pi)$. For any $\sigma \in \Sigma$, there exists $t(\sigma) \in \{1,...,T\}$ such that $\sigma \circ \sigma_{t(\sigma)} = \sigma_{Id}$. Thus $x \notin G(\pi) = G((\pi^{t(\sigma)})^{\sigma}) \Rightarrow \forall\sigma \in \Sigma$, $(x,\ldots,x) \notin G(\pi^{*\sigma})$ $\Rightarrow (x,\ldots,x) \notin \cup_{\sigma\in\Sigma} G(\pi^{*\sigma})$. Finally, suppose $F(\pi,\omega) \nsupseteq G(\pi,\omega)$ and let $y \in G(\pi) - F(\pi)$. Since there exists $\sigma' \in \Sigma$ such that $\sigma' \circ \sigma_1 = \sigma_{Id}$, then $(y, y_1,\ldots,y_{T-1}) \in G(\pi^{*\sigma'})$ for some $y_1,\ldots,y_{T-1} \in \omega$. Thus, $(y, y_1,\ldots,y_{T-1}) \in \cup_{\sigma\in\Sigma} G(\pi^{*\sigma}) - \varphi(\pi^*)$, which completes the proof $\square$

Finally, the proof of Theorem 1 involves situations with a very large number of alternatives. Obviously, there is a trade-off between practical applicability (which generally requires few alternatives) and theoretical concerns (which seek for results valid for as many voting rules as possible). Still there are contexts where the alternative set may be very large. Interpret $\omega^* = \mathbb{X}_{1\leq j\leq J}\omega_j$, where $\omega_j = \omega$ for all j, as a set of successive decisions to be made, where the set ω_t of all possible decisions at time t is the same for all periods t. Lexicographic preferences describes a time preference structure where two sequences of decisions are compared from the earliest period on which they differ. Note that we allow for preferences over ω to evolve over time. Within this context, separability is equivalent to time consistency of choice, meaning that it does not matter whether all decisions are taken at once or successive elections are held. Using Theorem 1, the strong paradox may hold even in cases where the set of time-wise possible decisions is small.

4 Separable voting rules and constitutions

How strong is the separability property? We investigate in this Part the existence of separable constitutions. An intermediate step is to consider the class of separable voting rules. We will focus on some Condorcet rules (section 4.1) and on scoring rules (section 4.2). We then show how separable constitutions can be built from separable voting rules (section 4.3).

Beforehand, we argue that separability can be linked to another property of voting rules called *composition-consistency* (Laffond, Lainé and Laslier (1996)). Suppose that the choice set ω is a finite union of H finite sets $\omega_1,...,\omega_h,...,\omega_H$. Moreover, suppose that preferences are such that each individual ranks through a linear order all such sets (in the sense that any element in some set ω_h is either preferred to or less preferred than, any element in another set $\omega_{h'}$). Similarly to the case of a lexicographic profile, each set is interpreted as a policy issue, while elements of this set are specific decision about this issue. Hence, individuals rank sets, and rank all elements of each set. A composition-consistent voting rule selects the best alternatives(s) in the best sets(s). Composition-consistency allows for sequentially computing winning sets, where best sets are chosen first, and second best alternatives are chosen within each of the best sets. This is formalized in the next two definitions.

Definition 8 *A profile π is decomposable on the choice set ω if there exists a partition $\theta = \{\omega_1, ..., \omega_H\}$ of ω into $H > 1$ non-empty subsets such that $\forall h \neq h' \in \{1, ..., H\}$, $\forall x, y \in \omega_h$, $\forall x', y' \in \omega_{h'}$, $x \ \pi_i \ x' \Leftrightarrow y \ \pi_i \ y'$.*

If π is decomposable on ω, then each preference π_i induces:
- a linear order $\pi_i \mid \theta$ over θ defined by: $\forall \ j, j' \in \{1, ..., H\}$, $j \ [\pi_i \mid \theta] \ j' \Leftrightarrow x \ \pi_i \ x'$ for at least one $x \in \omega_h$ and one $x' \in \omega_{h'}$
- H weak orders $\pi_i^1, ..., \pi_i^H$, where each π_i^h is the projection of π_i to ω_h

This leads to a profile $\pi \mid \theta$ over θ and to H profiles $\pi^1, ..., \pi^H$ respectively defined on $\omega_1, ..., \omega_H$.

Definition 9 *A voting rule F is composition-consistent if $F(\pi) = \cup_{h:\omega_h \in F(\pi|\theta)} F(\pi^h)$ for any decomposable profile π on ω.*

The next proposition states that composition-consistency together with neutrality imply separability.

Proposition 3 *A neutral and composition-consistent voting rule is separable.*

Proof Let $\omega = \mathbb{X}_{j=1}^J \omega_j$ and let π be a lexicographic profile on ω, where $1 \succ 2 \succ ... \succ J$. Note that π is decomposable on ω for the partition $\theta = \{\omega_x, x \in \omega_1\}$, where $\omega_x = \{x\} \times \mathbb{X}_{j=2}^J \omega_j$. If F is a neutral and composition-consistent voting rule, then $F(\pi) = \cup_{x:\omega_x \in \varphi(\pi/\theta)} F(\pi^x)$. Since π is lexicographic, it follows from neutrality that $F(\pi^x) = \{x\} \times F(\pi \mid \mathbb{X}_{j=2}^J \omega_j)$ for all $x \in \omega_1$. Thus $F(\pi) = \cup_{x:\omega_x \in F(\pi|\theta)} F(\pi^x) = \cup_{x:\omega_x \in F(\pi|\theta)} [\{x\} \times F(\pi \mid \mathbb{X}_{j=2}^J \omega_j)] = F(\pi^1) \times F(\pi \mid \mathbb{X}_{j=2}^J \omega_j)$. Iterating the argument for $j = 2, ..., J$ establishes separability $\square$

Note that the unanimity rule that selects all alternatives unanimously less preferred to no other alternative[6] is neutral and composition-consistent, hence separable. We show below that several well-known Condorcet voting rules, as well as the Borda rule, are separable.

4.1 Separable tournament solutions

We recall below the definition of several well-known tournament solutions. Given a tournament γ on some set ω, a path in γ is a vector $(x_1, \ldots, x_K) \in \omega^K$ of alternatives such that for each $k \in \{1, ..., K-1\}$, x_k defeats x_{k+1}. Given a path $(x_1, \ldots, x_K)$ in γ, we say that x_1 indirectly defeats x_K. Consider a tournament γ. The *Top-Cycle* of γ is the set $TC(\gamma)$ of all alternatives which directly or indirectly defeat all other alternatives. A path $(x_1, \ldots, x_K)$ in γ is transitive if $(x_k, x_{k'}) \in \gamma$ for any $1 \leq k < k' \leq K$. Moreover, a transitive path $(x_1, \ldots, x_K)$, is maximal if $(x, x_1, \ldots, x_K)$ is not transitive for any additional alternative x. The *Banks Set* of γ T is the set $B(\gamma)$ of all alternatives starting a maximal transitive path. The *Copeland Set* of γ is the set $C(\gamma)$ of alternatives which defeat the highest number of alternatives. Say that a covers b in γ (denoted by $a \gg_\gamma b$) if a defeats b and all alternatives that b defeats. The covering relation $\gg_\gamma$ is transitive, and the *Uncovered Set* of γ is the set $UC(\gamma)$ of elements of ω which are maximal for $\gg_\gamma$. A refinement of UC, denoted by UC^∞ is obtained by successively computing $UC(UC(\gamma))$, $UC(UC(UC((\gamma)))$, ... and so on until we no longer eliminate new alternatives. A subset ω' of ω is called a *covering set* of γ if it forms the

[6] Formally, the unanimity rule F^* is defined by $F^*(\pi) = \{x \in \omega : \forall y \in \omega - \{x\}$,there exists $i \in I$ such that $x \ \pi_i^+ \ y\}$.

Uncovered Set of the restriction of γ to ω', and if no additional alternative x is in the Uncovered Set of the restriction of γ to $\omega' \cup \{x\}$. The *Minimal Covering Set* is the (unique) minimal element $MC(\gamma)$ of $COV(\gamma)$ with respect to inclusion, where $COV(\gamma)$ is the family of covering sets of γ. The *Bipartisan Set* of T is the support $BP(\gamma)$ of the unique and symmetric equilibrium in mixed strategies of the 2-player zero-sum game where each of the players $1, 2$ have ω as set of pure strategies, and 1's payoffs are defined by: $p_1(x, y) = 1$ if $(x, y) \in T$, $= -1$ if $(y, x) \in T$, $= 0$ if $x = y$. defining the next solution, called Tournament Equilibrium Set (TEQ) requires more involved notations. Given any binary relation β defined on ω, $\omega' \subseteq \omega$ is called retentive for β if $(x, y) \in \beta \Rightarrow x \in \omega'$ for all $x \in \omega$ and all $y \in \omega'$. Moreover, the *Top Set* of β as the set $TS(\beta)$ of all subsets that are retentive for β and contain no retentive subset for β. TEQ is defined as the Top Set of a specific binary relation. Denote by $V(x, \gamma)$ the set of alternatives which defeat alternative x in γ. Given any tournament solution S, the binary relation $D_S(\gamma)$ is defined by $(x, y) \in D_S(\gamma)$ iff S chooses x from the restriction $\gamma \mid V(y, \gamma)$ of γ to $V(y, \gamma)$. The *Tournament Equilibrium Set* of γ is the (unique) subset $TEQ(\gamma) = TS(D_{TEQ}(\gamma))$.

Another well-known tournament solution is based on the Kemeny distance between tournaments. If β and β' are two tournaments over ω, the Kemeny distance between β and β' is the number $d(\beta, \beta')$ of pairs of alternatives β and β' disagree on. Formally, $d(\beta, \beta') = \{(x, y) \in \omega \times \omega : (x, y) \in \beta \text{ and } (y, x) \in \beta'\}$. A *Slater order* for tournament γ is a linear order β_{SL} over ω that is at minimal Kemeny distance to γ. The *Slater Set* of γ is the subset $SL(\gamma)$ of ω which are top-elements of Slater orders for γ.

It is shown in Laffond et al. (1996) that UC, COV, B, BP, UC^{∞} and TEQ are composition-consistent. Since a tournament solution defines a neutral voting rule, Proposition 3 ensures that all five solutions above are separable. Furthermore, neither TC nor C nor SL is composition-consistent. The next Proposition (proved in the Appendix) shows that nonetheless C and SL are separable.

Proposition 4 *(1) TC is not separable, (2) C is separable, and (3) SL is separable.*

4.2 Separable scoring rules

We turn to scoring rules. The next definition formalizes truncated scoring rules.

Definition 10 *A scoring rule is $h-$truncated if for any $\omega \in \Delta$, for any profile π on ω with $I(\pi) > h$, the score vector s for $I(\pi)$ is such that $s_1 > s_2 > ... > s_h = ... = s_{I(\pi)}$.*

An $h-$truncated scoring rule is such that all scores less than the h^{th} highest one coincide. Put differently, all individuals give the same score to all alternatives above a certain rank in their weak order. Note that the 2-truncated Borda rule is the plurality rule, where each most preferred alternative is given a score 1, and all other alternatives are given a score 0.

The Borda rule is not composition-consistent but separable. However no truncated scoring rule is separable. This is stated in the next proposition.

Proposition 5 *(1) The Borda rule is separable, and (2) for any integer $h > 1$, no $h-$truncated scoring rule is separable.*

4.3 From separable voting rules to separable constitutions

It is easy to see that a constitution based on a separable voting rule may not be separable. For an illustration, pick any separable tournament solution F, and consider $\omega = \omega_1 \times \omega_2$ where $\omega_1 = \{a, b, c\}$ and $\omega_2 = \{x, y\}$. Moreover, suppose that all individuals in $N = \{1, 2\}$ have lexicographic preferences over ω defined in the following profile π:

$$\pi = \begin{pmatrix} 1 & 2 \\ \hline (a,x) & (c,x) \\ (a,y) & (c,y) \\ (b,x) & (b,x) \\ (b,y) & (b,y) \\ (c,x) & (a,x) \\ (c,y) & (a,y) \end{pmatrix}$$

Define the representation function δ by: $\forall N, \omega \in \Delta$, $\forall \pi \in \mathcal{R}(\omega)^N$, $\forall D = \{D_1, ..., D_T\} \in \mathcal{P}(N)$, $\forall t \in \{1, ..., T\}$, $\delta_t(\pi_{D_t}) = \pi_1$ if $|\omega|$ is even, and $\delta_t(\pi_{D_t}) = -\pi_1$ if $|\omega|$ is odd. Now let $D = \{N\} = \{\{1,2\}\}$. Since $|N|$ is even, $\delta(\pi, D\} = \pi_1$, and this implies $F_\delta(\pi) = \{a, x\}$. Moreover, since $|\omega_1|$ is odd, we get $F_\delta(\pi^1) = c$, which proves that F_δ is not separable.

This simple example shows how the translation of voters' preferences into representative preferences may prevent to preserve separability. However, it is straightforward to check that every constitution based on a separable scoring rule is separable (the formal proof is left to the reader). Furthermore, the same result holds for constitutions based on separable tournament solutions, as stated in Proposition 6 below (see Appendix for the proof).

Proposition 6 *A constitution based on a separable tournament solution is separable.*

As an immediate corollary of Theorem 1, we get that every constitution based either on a separable tournament solution or on the Borda rule is exposed to the strong paradox iff it is exposed to the paradox.

5 Strong paradox for constitutions based on scoring rules

We have shown the equivalence between the referendum paradox and the strong referendum paradox for constitutions based on several tournament solutions or on the Borda rule. Since truncated scoring rules are not separable, Theorem 1 does not apply to constitutions based on such rules. Nonetheless, we show below that every constitution (sequential or not) based on a scoring rule is exposed to the strong paradox. Moreover, this result holds even when imposing a partition in (resp. almost) two equally sized districts (resp. for sequential constitutions). Say that a partition $D = \{D_1, ..., D_T\}$ of the voter set N is fair if the distribution of district sizes is the closest to equality.[7]

Theorem 2 *(1) Let F_δ be a constitution based on some scoring rule F. There exist a choice set ω, a set N of voters, a profile π over ω and a fair partition D of N such that $F(\pi) \neq F_{D,\delta}(\pi^\sigma)$ for all permutations σ of N.*

(2) Let F_δ be a sequential constitution based on some scoring rule F. There exist a choice set ω, a set N of voters , a profile π over ω and a fair partition D of N such that $F(\pi) \neq F_{D,\delta}(\pi^\sigma) \mid \cup_{1 \le t \le T} F(\pi^\sigma_{D_t})$ for all permutations σ of N.

6 Further comments

We present in this paper a general model of two-step elections where voters submit preferences rather than a single vote, and where several winning alternatives may be chosen. We show that if the indirect election procedure is separable, exposure to the referendum paradox is equivalent to exposure to the strong

[7] Formally, a partition $D = \{D_1, ..., D_T\}$ of N is fair if there is no other partition $D' = \{D'_1, ..., D'_{T'}\}$ of N such that $MAX_{1 \leqslant t,t' \leqslant T'}[\frac{|D_t|}{|D_{t'}|}] < MAX_{1 \leqslant t,t' \leqslant T}[\frac{|D_t|}{|D_{t'}|}]$.

referendum paradox: direct and indirect elections give mutually inconsistent results for some distribution of voters across districts if and only if they do so for any distribution of voters across districts. Hence, the potential inconsistency between direct democracy and representative democracy pertains to a logical inconsistency and as such, does not relate to the standard argument of gerrymandering. To our knowledge, the characterization of voting rules immune to the strong paradox remains to be done. As long as non-dictatorship is required, this class is likely to be very narrow (if not empty). Indeed, separability is not a restrictive property, since it holds for many tournament solutions. Moreover, separability is not necessary to the strong referendum paradox. Indeed, we prove that every indirect election based on a scoring rule is exposed to the strong paradox.

Finally, immunity to the strong referendum paradox may be compared to the consistency property introduced in Smith (1973) and Young (1975). Consistency states that, whenever two disjoint electorates facing the same choice set choose the same winning set, then the winning set for the aggregated electorate should be this winning set. It is already known that no Condorcet voting rule is consistent, whereas all scoring rules are. Immunity to the strong referendum paradox actually follows the reverse direction: a voting rule is immune to the strong paradox if a single electorate can be divided into disjoint sub-electorates so that the initial choice is confirmed when completed from separate sub-electorate profiles. One consequence of our results is that immunity to the strong paradox no longer provides a clear-cut distinction between Condorcet rules and scoring rules.

7 References

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8 Proofs

8.1 Proof of Proposition 4

Proof of assertion (1): Let $\omega = \omega_1 \times \omega_2 \in \Delta$ and pick up a lexicographic profile π on ω, with $1 \succ 2$. Let $\gamma_1 = \gamma(\pi^1)$ and $\gamma_2 = \gamma(\pi^2)$. Consider $x_1, x_1' \in TC(\gamma_1)$, $y_2 \in TC(\gamma_2)$, $y_2' \in \omega_2 - TC(\gamma_2)$, with $(x_1', x_1) \in \gamma_1$. Separability implies that $z = (x_1, y_2) \in TC(\gamma(\pi))$ and $z' = (x_1', y_2') \notin TC(\gamma(\pi))$. However, since π is lexicographic with $1 \succ 2$, then $(z', z) \in \gamma(\pi)$, which contradicts $z' \notin TC(\gamma(\pi))$. Thus TC is not separable.

Proof of assertion (2): Given a tournament γ on ω, define for each $x \in \omega$ the Copeland score of x in γ as $c(x, \gamma) = |\{y \in \omega : (x, y) \in \gamma\}|$. Thus $C(\gamma) = \{x \in \omega : c(x, \gamma) \geq c(y, \gamma), \forall y \in \omega\}$. Pick $x = (x_1, x_2) \in \omega_1 \times \omega_2$, and denote by $c(x_h, \gamma_h)$ the Copeland score of x_h in $\gamma_h = \gamma(\pi^h)$, where $h = 1, 2$. Let $x^* = (x_1^*, x_2^*) \in C(\gamma)$. Since π is lexicographic, then $c(x^*, \gamma) = c(x_1^*, \gamma_1).\,|\omega_2| + c(x_2^*, \gamma_2)$. Moreover, since $c(x^*, \gamma) \geq c((x_1^*, x_2), \gamma)$ for all $x_2 \in \omega_2$, it must be true that $c(x_2^*, \gamma_2) \geq c(x_2, \gamma_2)\ \forall\ x_2 \in \omega_2$, hence that $x_2^* \in C(\gamma_2)$. Similarly, $[c(x^*, \gamma) \geq c((x_1, x_2^*), \gamma)$ for all $x_1 \in \omega_1] \Rightarrow [x_1^* \in C(\gamma_1)]$. Thus, $C(\gamma) \subseteq C(\gamma_1) \times C(\gamma_2)$. Finally, if $x = (x_1, x_2) \in C(\gamma_1) \times C(\gamma_2)$, then $c(x, \gamma) = c(x^*, \gamma)$. Hence, $C(\gamma) = C(\gamma_1) \times C(\gamma_2)$. If $\omega = \mathbb{X}_{j=1}^J \omega_j$, separability is obtained by replicating the same argument for $j = 3, ..., J$.

Proof of assertion (3): Let $\omega = \omega_1 \times \omega_2$ and let π be lexicographic on ω with $1 \succ 2$. Given a Slater order $\succ$, denote by $h(\succ)$ the alternative with rank $h \in \{1, ..., |\omega|\}$ in $\succ$. Since π is lexicographic, there exists a partition $Z = \cup_{1 \leqslant t \leqslant |\omega_1|} Z_t$ of ω, where $Z_t = \{x_t\} \times \omega_2$ and $t = 1, ..., |\omega_1|$, such that $\forall t, t'$, $\forall (x, x') \in Z_t \times Z_{t'}$, $(x, x') \in \gamma(\pi) \Rightarrow (y, y') \in \gamma(\pi)$ for all $(y, y') \in Z_t \times Z_{t'}$. Each Z_t is called a component of $\gamma(\pi)$. Let $x^* = (x_1^*, x_2^*) \in SL(\gamma(\pi))$.

We claim that $\succ$ must be such that $\forall\ Z_t \in Z$, $\forall x, x' \in Z_t$, $[x = h(\succ)$, $x' = h'(\succ)$ and $h < h']$ $\Rightarrow [\{y \in \omega : \exists k \in \{h+1, ..., h'-1\}$ s.t. $y = k(\succ)\} \subset Z_t]$. In words, $\succ$ is such that all components Z_t are intervals for $\succ$. Indeed, let $x, y \in \omega$ such that $x_t = y_t$ (hence x, y belong to the same component Z_t) and $x = h(\succ)$, $y = h'(\succ)$, with $h < h'$. Let $A = \{z \in \omega - Z_t : (z, x) \in \gamma(\pi)$ and $z = k(\succ)$ for some $h < k < h'\}$, $B = \{z \in \omega - Z_t : (x, z) \in \gamma(\pi)$ and $z = k(\succ)$ for some $h < k < h'\}$, and $C = \{z \in Z_t : z = k(\succ)$ for some $h \leq k \leq h'\}$. Define the linear order $\succ^*$ over ω by: (1) $\forall z \notin A \cup B \cup C$, $\forall z' \in \omega$, $z \succ z' \Leftrightarrow z \succ^* z'$, (2) $\forall z \in A$, $\forall z' \in C \cup B$, $z \succ^* z'$, (3) $\forall z \in C$, $\forall z' \in B$, $z \succ^* z'$, (4) $\forall (z, z') \in A^2 \cup B^2 \cup C^2$, $z \succ z' \Leftrightarrow z \succ^* z'$. Thus, $\succ^*$is a reshuffling of all alternatives in $\{h(\succ), ..., h'(\succ)\}$ which is consistent with partition Z. It is obviously seen that if $A \cup B \neq \varnothing$, then $d(T, \succ^*) < d(T, \succ)$, which contradicts that $\succ$ is a Slater order for $\gamma(\pi)$.

If $\omega' \subset \omega$, denote by $SL(\gamma(\pi) \mid \omega')$ the Slater set of the tournament restricted to ω'. It is easy to prove that the restriction of $\succ$ to any component Z_t is a Slater order for $\gamma(\pi) \mid Z_t$. This implies that $x_2^* \in SL(\gamma_2)$, where $\gamma_2 = \gamma(\pi^2)$.

Now, pick any linear order θ_1 over ω_1, and let θ_2 be a Slater order for γ_2. Define the linear order μ on ω by : $\forall z = (z_1, z_2) \in \omega$, $\forall z' = (z_1', z_2') \in \omega$, $[z, z' \in Z_t] \Rightarrow [(z, z') \in \mu \Leftrightarrow (z_2, z_2') \in \theta_2]$, and $[z \in \delta_t, z' \in \delta_{t'}, t \neq t'] \Rightarrow [(z, z') \in \mu \Leftrightarrow (z_1, z_1') \in \theta_1]$. Since all components Z_t have the same cardinality, it follows that $d(\gamma(\pi), \mu) = |\omega_1|\,.d(\gamma_2, \theta_2) + |\omega_2|\,.d(\gamma_1, \theta_1)$, where $\gamma_1 = \gamma(\pi^1)$. Note that the first term of the sum is a constant. This implies that if μ is a Slater order for $\gamma(\pi)$, then θ_1 is a Slater order for γ_1 (otherwise, there would exist a linear order θ_1' on ω_1 such that $d(\gamma(\pi), \mu') < \Delta(\gamma(\pi), \mu)$, where μ' is defined as μ once θ_1 being replaced with θ_1'). Thus $x_1^* \in SL(\gamma_1)$. Since $x_2^* \in SL(\gamma_2)$, we have proved that $SL(\gamma(\pi)) \subseteq SL(\gamma_1) \times SL(\gamma_2)$.

Suppose finally that θ_1' is some Slater order for γ_1 with x_1' at top, and that θ_2' is some Slater order for γ_2 with x_2' at top. Let μ' be the linear order on ω defined as above with θ_1' and θ_2' instead of θ_1 and θ_2. It follows from the construction of μ' that $d(\gamma(\pi), \mu) = d(\gamma(\pi), \mu')$, which proves that μ' is a Slater order for $\gamma(\pi)$. Thus $SL(\gamma_1) \times SL(\gamma_2) \subseteq SL(\gamma(\pi))$. The generalization to any Cartesian product $\omega = \mathbb{X}_{j=1}^J \omega_j$ is straightforward.

8.2 Proof of Proposition 5

Proof of assertion (1): Let F be the Borda rule. Pick a profile π lexicographic on $\omega = \omega_1 \times \omega_2$, with $1 \succ 2$ and $I(\pi) = z$. For each voter i, let $a_i = I(\pi_i^2)$ be the number of indifference classes in π_i^2. It is easily seen that for any i and any $x = (x_1, x_2) \in \omega$, one has that $s_{rank_{\pi_i}}(x) = z - \Big[a_i.(rank_{\pi_i^1}(x_1) - 1) + rank_{\pi_i^2}(x_2)\Big]$. Hence,

$x^* = (x_1^*, x_2^*) \in \varphi(\pi^1) \times \varphi(\pi^2) \Leftrightarrow$
$\forall x_1 \in \omega_1 \ , \forall x_2 \in \omega_2 \ , \ \ S^{\pi^1}(x_1^*) \geq S^{\pi^1}(x_1)$ and $S^{\pi^2}(x_2^*) \geq S^{\pi^2}(x_2) \quad \Leftrightarrow$
$\sum\limits_{i \in I} rank_{\pi_i^1}(x_1^*) \leq \sum\limits_{i \in I} rank_{\pi_i^1}(x_1)$ and $\sum\limits_{i \in I} rank_{\pi_i^2}(x_2^*) \leq \sum\limits_{i \in I} rank_{\pi_i^2}(x_2) \Leftrightarrow$
$\sum\limits_{i \in I} \{z - \Big[a_i.(rank_{\pi_i^1}(x_1^*) - 1) + rank_{\pi_i^2}(x_2^*)\Big]\} \geq \sum\limits_{i \in I} \{z - \Big[a_i.(rank_{\pi_i^1}(x_1) - 1) + rank_{\pi_i^2}(x_2)\Big] \Leftrightarrow$
$S^{\pi}(x_1^*, x_2^*) \geq S^{\pi}(x_1, x_2) \Leftrightarrow x^* = (x_1^*, x_2^*) \in \varphi(\pi)$. Thus, the Borda rule is separable.

Proof of assertion (2): Consider a 2$-$truncated scoring rule F. Let $\omega = \omega_1 \times \omega_2$, where $\omega_1 = \{x_1, x_2, x_3\}$ and $\omega_2 = \{y_1, y_2, y_3\}$. Let $N = \{1, 2, 3\}$, and let π be the lexicographic profile defined on ω by $1 \succ 2$ and:

$$\pi^1 = \begin{pmatrix} 1 & 2 & 3 \\ \hline x_1 & x_1 & x_2 \\ x_2 & x_2 & x_1 \\ x_3 & x_3 & x_3 \end{pmatrix} \pi^2 = \begin{pmatrix} 1 & 2 & 3 \\ \hline y_2 & y_1 & y_1 \\ y_1 & y_2 & y_2 \\ y_3 & y_3 & y_3 \end{pmatrix}$$

Let $s = (s_1, s_2, \ldots, s_9)$ (resp. $s' = (s'_1, s'_2, s'_3)$) be the score vector for $I(\pi)$ (resp. for $I(\pi^1)$ and $I(\pi^2)$). Note that we can write w.l.o.g. that $s_1 > s_2 = \ldots = s_9 = 0$ and $s'_1 > s'_2 = s'_3 = 0$. Then $S^{\pi^1}(x_1) = 2s'_1 > S^{\pi^1}(x_2) = s'_1 \Rightarrow x_2 \notin F(\pi^1)$. Furthermore, $S^{\pi}((x_2, y_1)) = s_1 \geq S^{\pi}(x) \ \forall x \in \omega \Rightarrow (x_2, y_1) \in F(\pi)$. Thus $F(\pi) \neq F(\pi^1) \times F(\pi^2)$.

Now consider a $h-$truncated scoring rule φ with $h \geq 3$. Let $\omega = \omega_1 \times \omega_2$, where $\omega_1 = \{x_1, x_2, x_3, \ldots, x_{h+1}\}$ and $\omega_2 = \{y_1, y_2, y_3 \ldots, y_{h+1}\}$. Let $N = \{1, 2, 3\}$, and let π be the lexicographic profile defined on ω by $1 \succ 2$ and:

$$\pi^1 = \begin{pmatrix} 1 & 2 & 3 \\ \hline x_2 & x_3 & x_1 \\ x_1 & x_1 & x_3 \\ \vdots & x_2 & \vdots \\ x_3 & \vdots & x_2 \end{pmatrix} \quad \pi^2 = \begin{pmatrix} 1 & 2 & 3 \\ \hline y_2 & y_3 & y_1 \\ y_1 & y_1 & y_3 \\ \vdots & y_2 & \vdots \\ y_3 & \vdots & y_2 \end{pmatrix}$$

Let $s = (s_1, s_2, \ldots, s_{(h+1)(h+1)})$ (resp. $s' = (s'_1, s'_2, \ldots, s'_{(h+1)})$) be the score vector for $I(\pi)$ (resp. for $I(\pi^1)$ and $I(\pi^2)$), with $s_1 > \ldots > s_h = \ldots = s_{(h+1)(h+1)} = 0$ (resp. $s'_1 > \ldots > s'_h = s'_{h+1} = 0$). Then $S^{\pi^1}(x_1) = s'_1 + 2s'_2 > s'_1 + s'_3 = S^{\pi^1}(x_2)$ implies that $x_2 \notin F(\pi^1)$. Finally, since $S^{\pi}((x_2, y_2)) = s_1 \geq S^{\pi}(x)$ $\forall x \in \omega$, we get $F(\pi) \neq F(\pi^1) \times F(\pi^2)$ which completes the proof.

8.3 Proof of Proposition 6

We first recall that a constitution F_δ is based on a tournament solution F if for all $N, \omega \in \varDelta$, for all $D = \{D_1, ..., D_T\} \in \mathcal{P}(N)$, and for all $\pi \in \mathcal{L}(\omega)^N$ such that $\gamma(\pi)$ and all $\gamma(\pi_{D_t})$ are well-defined, we have $\delta_t(\pi, D_t) = \gamma(\pi_{D_t})$ for all t, $\delta(\pi, D) = \gamma(\gamma(\pi_{D_1}), ..., \gamma(\pi_{D_T}))$, and $F_\delta(\pi) = F(\delta(\pi, D))$. Pick a separable tournament solution F. Let π be a lexicographic profile over $\omega_1 \times \omega_2$ with $1 \succ 2$ and let π^1 and π^2 be the induced profiles respectively on ω_1 and on ω_2 (the proof easily extends to higher-dimension choice sets). Then we must prove that $F(\gamma(\gamma(\pi_{D_1}),...,\gamma(\pi_{D_T}))) = F(\gamma(\gamma(\pi^1_{D_1}),...,\gamma(\pi^1_{D_T}))) \times F(\gamma(\gamma(\pi^2_{D_1}),...,\gamma(\pi^2_{D_T})))$.

Since F is separable, it suffices to prove that there exists a lexicographic profile $\widetilde{\pi}$ on ω such that $\gamma(\widetilde{\pi}) = \gamma(\gamma(\pi_{D_1}),...,\gamma(\pi_{D_T}))$, with $\gamma(\widetilde{\pi}^1) = \gamma(\gamma(\pi^1_{D_1}),...,\gamma(\pi^1_{D_T}))$ and $\gamma(\widetilde{\pi}^2) = \gamma(\gamma(\pi^2_{D_1}),...,\gamma(\pi^2_{D_T}))$.

From Mc Garvey's theorem (Mc Garvey (1953), see Laslier (1997), page 35 for a proof), there exists a profile $\widehat{\pi}^1$ over ω_1 with N_1 as set of voters such that $\gamma(\widehat{\pi}^1) = \gamma(\gamma(\pi^1_{D_1}),...,\gamma(\pi^1_{D_T}))$ and there exists a profile $\widehat{\pi}^2$ over ω_2 with N_2 as set of voters such that $\gamma(\widehat{\pi}^2) = \gamma(\gamma(\pi^2_{D_1}),...,\gamma(\pi^2_{D_T}))$. Mc Garvey's construction of $\widehat{\pi}^1$ is based on associating two voters with each of the $\frac{|\omega_1|.|\omega_1-1|}{2}$ pairwise comparisons in $\gamma(\gamma(\pi^1_{D_1}),...,\gamma(\pi^1_{D_T}))$. Using the same construction for $\widehat{\pi}^2$, we can suppose that N_1 and N_2 have an even cardinality.

Suppose without loss of generality that $N_2 > N_1$. Since $|N_2| - |N_1|$ is even, we can complete $\widehat{\pi}^1$ by adding pairs of mutually inverse orders so as to get the same tournament with N_2 voters. Denote by $\overline{\pi}^1$ this new profile. Then define $\widetilde{\pi}$ involving N_2 as set of voters by: $\widetilde{\pi}^1 = \overline{\pi}^1$, $\widetilde{\pi}^2 = \widehat{\pi}^2$ and $\forall(x_1,x_2),(y_1,y_2) \in \omega_1 \times \omega_2$, $\forall i \in N_2$, $(x_1,x_2)\ \widetilde{\pi}_i\ (y_1,y_2)$ if either $x_1\ \widetilde{\pi}^1_i\ y_1$ or $x_1 = y_1$ and $x_2\ \widetilde{\pi}^2_i\ y_2$. Since $\widetilde{\pi}$ is lexicographic for all $i \in N_2$. Then $\widetilde{\pi}$ is lexicographic and $\gamma(\widetilde{\pi}) = \gamma(\gamma(\pi_{D_1}),...,\gamma(\pi_{D_T}))$, $\gamma(\widetilde{\pi}^1) = \gamma(\gamma(\pi^1_{D_1}),...,\gamma(\pi^1_{D_T}))$ and $\gamma(\widetilde{\pi}^2) = \gamma(\gamma(\pi^2_{D_1}),...,\gamma(\pi^2_{D_T}))$. $\gamma(\widetilde{\pi}) = \gamma(\gamma(\pi_{D_1}),...,\gamma(\pi_{D_T}))$, the proof is complete.

8.4 Proof of Theorem 2

We establish the existence of the strong paradox by proving the following: for any scoring rule F, there exist $\omega, N \in \Delta$ and $\pi \in \mathcal{R}(\omega)^N$ such that $F(\pi) \neq F_{D,\delta}(\pi)$ for any fair partition D of N.

Proof of assertion (1): Consider $\omega = \Theta \cup \Phi$, where Θ (resp. Φ) is the set of all one-to-one mappings from $\{1,1',2,2'\}$ (resp. $\{1,1',1'',3\}$) to the set of voters $N = \{1,2,3,4\}$. Pick any scoring rule F with score vector $s = (s_1,s_2,s_3)$ for profiles π with $I(\pi)$, where w.l.o.g. $s_3 = 0$.

Suppose first that $s_1 > s_2$.

Define profile π on ω sas follows. For any $i \in N$, for any $x \in \Theta$, $Rank_{\pi_i}(x) = r$ if and only if $x^{-1}(i) \in \{r,r'\}$, where $r = 1,2$. Similarly, for any $x \in \Phi$, $Rank_{\pi_i}(x) = r$ if and only if $x^{-1}(i) \in \{r,r',r''\}$, where $r = 1,3$. Since there are 4 voters, any fair partition must be of the form $D = \{D_1,D_2\}$, where $D_1 = \{i.i'\}$ and $D_2 = \{j,j'\}$. Table 5 describes preferences in district D_1, each associated with a specific set of ranks given by i and i' to $x \in \omega$. It follows from preferences that Θ can be partitioned into 3 non-empty subsets $\Theta_1,\Theta_2,\Theta_3$, while Φ is partitioned into two non-empty subsets Φ_1,Φ_2, each subset being associated with a set of ranks: for instance, any $x \in \Theta_1$ is such that both voters in D_1 rank x first. The third line indicates the total score obtained (from voters in D_1) by each alternative x, while the fourth line gives the rank of x given by the representative of D_1 (under assumption $s_1 > s_2 \neq 0$).

Table 5

$x \in$	Θ_1	Θ_2	Θ_3	Φ_1	Φ_2
$(Rank_{\pi_i}(x), Rank_{\pi_{i'}}(x))$	$(1,1)$	$(1,2)$	$(2,2)$	$(1,1)$	$(1,3)$
$s_{rank_{\pi_i}}(x) + s_{rank_{\pi_{i'}}}(x)$	$2s_1$	$s_1{+}s_2$	$2s_2$	$2s_1$	s_1
$Rank_{\widetilde{F}(\pi_{D_1})}(x)$	1	2	3 or 4	1	3 or 4

Preferences in D_2 are depicted in Table 6 along the same lines as above.

Table 6

$x \in$	Θ_1	Θ_2	Θ_3	Φ_1	Φ_2
$(Rank_{\pi_j}(x), Rank_{\pi_{j'}}(x))$	$(2,2)$	$(1,2)$	$(1,1)$	$(1,3)$	$(1,1)$
$s_{rank_{\pi_j}}(x) + s_{rank_{\pi_{j'}}}(x)$	$2s_2$	$s_1{+}s_2$	$2s_1$	s_1	$2s_1$
$Rank_{\widetilde{F}(\pi_{D_2})}(x)$	3 or 4	2	1	3 or 4	1

Note that, since nothing allows to compare s_1 and $2s_2$, indeterminacy occurs about the rank of all $x \in \Theta_3 \cup \Phi_2$ in the preference $\widetilde{F}(\pi_{D_1})$ of D_1 representative (the same for all $x \in \Theta_1 \cup \Phi_1$ in D_2 representative's preference). Moreover, $F(\pi) = F_{D,\delta}(\pi)$ requires $s_2 > 0$. Indeed, if $s_1 > s_2 = s_3 = 0$, then $F(\pi) = \Phi$. Let $\delta(\pi, D) = (\widetilde{F}(\pi_{D_1}), \widetilde{F}(\pi_{D_2}))$ be the representative profile. Since $I(\delta(\pi, D)) = 3$ it is easily checked that $S^{\delta(\pi,D)}(x) = S^{\delta(\pi,D)}(y) = s_1$ for all $(x, y) \in \Theta_1 \times \Phi_1$. Then $\Phi_1 \subset F_{D,\delta}(\pi)$ implies $\Theta_1 \subset F_{D,\delta}(\pi)$, and therefore $F(\pi) \neq F_{D,\delta}(\pi)$.

Then consider the following cases:

Case 1: $s_1 < 2s_2$

It is easily checked that $F(\pi) = \Theta$. Moreover, $I(\delta(\pi, D)) = 4$. Let $\overline{s}$ be the score vector for $\delta(\pi, D)$. Then $F(\pi) \neq F_{D,\delta}(\pi)$ requires $\overline{s}_1 + \overline{s}_3 = 2\overline{s}_2$ (otherwise either $\Theta_1 \cap F_{D,\delta}(\pi) = \varnothing$ or $\Theta_2 \cap F_{D,\delta}(\pi) = \varnothing$). Consider profile π' defined on $\omega' = \{x_1, x_2, x_3, x_4\}$ and $\widetilde{\pi}'$defined on $\omega'' = \{x_1, x_2, x_3, x_4, x_5\}$ by:

$$\pi' = \begin{pmatrix} 1,2,3 & 4 \\ \hline x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_1 \\ x_4 & x_4 \end{pmatrix} \quad \widetilde{\pi}' = \begin{pmatrix} 1,2,3,4 & 5,6 \\ \hline x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_4 \\ x_4, x_5 & x_1, x_5 \end{pmatrix}$$

Suppose that $\overline{s}_1 > \overline{s}_2$. Since $\overline{s}_1 + \overline{s}_3 = 2\overline{s}_2$, then $2\overline{s}_1 + \overline{s}_3 = 3\overline{s}_2$. Thus $S^{\pi'}(x_1) = 3\overline{s}_1 + \overline{s}_3 > S^{\pi'}(x_2) = \overline{s}_1 + 3\overline{s}_2 > S^{\pi'}(x_3) = \overline{s}_2 + 3\overline{s}_3 > S^{\pi'}(x_4) = 4\overline{s}_4$, so that $\varphi(\pi') = \{x_1\}$. There are 3 fair partitions of $N = \{1, 2, 3, 4\}$: $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$. If $D = \{D_1, D_2\} = \{\{1, 2\}, \{3, 4\}\}$, then $S^{\widetilde{F}(\pi'_{D_1})}(x_1) = \overline{s}_1 = S^{\widetilde{F}(\pi'_{D_2})}(x_2)$, $S^{\widetilde{F}(\pi'_{D_1})}(x_2) = S^{\widetilde{F}(\pi'_{D_2})}(x_1) = \overline{s}_2$, $S^{\widetilde{F}(\pi'_{D_1})}(x_3) = S^{\widetilde{F}(\pi'_{D_2})}(x_3) = \overline{s}_3$, and $S^{\widetilde{F}(\pi'_{D_1})}(x_3) = S^{\widetilde{F}(\pi'_{D_2})}(x_3) = \overline{s}_4$. It follows that $F_{D,\delta}(\pi') = \{x_1, x_2\} \neq F(\pi')$. The same argument holds for the other two fair apportionments. Suppose that $\overline{s}_1 = \overline{s}_2 = \overline{s}_3$. Then $F(\widetilde{\pi}') = \{x_2, x_3\}$. If $D = \{D_1, D_2\} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$ is the fair partition of $N = \{1, 2, 3, 4, 5\}$, then $S^{\widetilde{F}(\widetilde{\pi}'_{D_1})}(x_h) = \overline{s}_1$, $1 \leq h \leq 5$, while $S^{\widetilde{F}(\widetilde{\pi}'_{D_2})}(x_h) = \overline{s}_1$, $1 \leq h \leq 4$. Thus herefore $F_{D,\delta}(\widetilde{\pi}') = \{x_1, x_2, x_3, x_4\} \neq F(\widetilde{\pi}')$. If $D = \{D_1, D_2\} = \{\{1, 2, 5\}, \{3, 4, 6\}\}$ is the fair partition of N, then $S^{\widetilde{F}(\widetilde{\pi}'_{D_1})}(x_h) = S^{\widetilde{F}(\widetilde{\pi}'_{D_2})}(x_h) = \overline{s}_1$, $1 \leq h \leq 4$, leading to the same conclusion. Hence $F_{D,\delta}(\widetilde{\pi}') \neq \varphi(\widetilde{\pi}')$ for any fair partition D.

Case 2: $s_1 = 2s_2$

Since $s_1 > s_2$ we directly get that $F_{D,\delta}(\pi'_{\omega''}) \neq \varphi(\pi'_{\omega''})$, where $\omega'' = \{x_1, x_2, x_3\}$.

Case 3: $s_1 > 2s_2 > 0$

Consider profile π'' defined on ω'' by

$$\pi'' = \begin{pmatrix} 1,2 & 3 & 4 \\ \hline x_1 & x_1, x_2 & x_2 \\ x_2 & x_3 & x_3 \\ x_3 & & x_1 \end{pmatrix}$$

Since $S^{\pi''}(x_1) = 3s_1$, $S^{\pi''}(x_2) = 2s_1 + 2s_2$, and $S^{\pi''}(x_3) = 2s_2$, then $s_1 > 2s_2$ implies that $F(\pi) = \{x_1\}$. If $D = \{D_1, D_2\} = \{\{1, 2\}, \{3, 4\}\}$, then $S^{\widetilde{F}(\pi_{D_1})}(x_1) = S^{\widetilde{F}(\pi_{D_2})}(x_2) = s_1$, $S^{\widetilde{F}(\pi''_{D_1})}(x_2) = S^{\widetilde{F}(\pi''_{D_2})}(x_1) = s_2$ and $S^{\widetilde{F}(\pi''_{D_1})}(x_3) = S^{\widetilde{F}(\pi''_{D_2})}(x_3) = 0$. It follows that $F_{D,\delta}(\pi'') = \{x_1, x_2\} \neq F(\pi'')$. It is straightforward to check that the same conclusion holds for the other two fair partitions.

Thus, we have proved that assertion (1) holds for any score vector s for 3 indifference classes such that $s_1 > s_2$.

It remains to consider score vectors with $s_1 = s_2$. Let $N = \{1, ..., 5\}$. Any fair partition of N must be of type $D = \{D_1, D_2\}$, where $|D_1| = 2$ and $|D_2| = 3$. Using the same type of construction as above, let $\omega = \Theta \cup \Phi$, where Θ (resp. Φ) is now the set of one-to-one mappings from $A_\Theta = \{1, 1', 2, 3, 3'\}$ (resp. $B_\Phi = \{1, 1', 1'', 2, 3\}$) to N. Preferences are defined in the same way as in the first example of the proof: $\forall x \in \omega$, $\forall i \in I$, $Rank_{\pi_i}(x) = r$ if and only if $x^{-1}(i) \in \{r, r', r''\}$. It follows from the definition of A_Θ and B_Φ that, given any x in Θ (resp. in Φ), the number $\alpha(x)$ of voters i with $x^{-1}(i) \in \{3, 3'\}$ is 2 (resp. 1).

Denote by $\alpha_j(x)$ the number of i in district D_j, $j = 1, 2$, with $x^{-1}(i) \in \{3, 3'\}$. Then Θ is partitioned into sets Θ_0, Θ_1 and Θ_2 with, for all $h = 0, 1, 2$, $x \in \Theta_h$ if and only if $(\alpha_1(x), \alpha_2(x)) = (h, 2-h)$. Similarly, Φ is partitioned into sets Φ_0 and Φ_1 with, for $h = 0, 1$, $x \in \Phi_h$ if and only if $(\alpha_1(x), \alpha_2(x)) = (h, 1-h)$. Table 7 gives district preference in D_1 and D_2:

Table 7

$x \in$	Θ_0	Θ_1	Θ_2	Φ_0	Φ_1
$(\alpha_1(x), \alpha_2(x))$	$(0,2)$	$(1,1)$	$(2,0)$	$(0,1)$	$(1,0)$
$\sum_{i\in D_1} s_{rank_{\pi_i}}(x)$	$2s_1$	s_1	0	$2s_1$	s_1
$\sum_{i\in D_2} s_{rank_{\pi_i}}(x)$	s_1	$2s_1$	$3s_1$	$2s_1$	$3s_1$
$Rank_{\widetilde{F}(\pi_{D_1})}(x)$	1	2	3	1	2
$Rank_{\widetilde{F}(\pi_{D_2})}(x)$	3	2	1	2	1

Since $\forall x \in \Theta$, $\forall x' \in \Phi$, $S^{\pi}(x) = 3s_1 < 4s_1 = S^{\pi}(x')$ then $F(\pi) = \Phi$. Furthermore since $I(\delta(\pi, D)) = 3$, then $\forall x \in \Phi \cup \Theta_1$, $\forall y \in \Theta_0 \cup \Theta_2$, $S^{\delta(\pi,D)}(x) = 2s_1 > s_1 = S^{\delta(\pi,D)}(y)$ so $F_{D,\delta}(\pi) = \Phi \cup \Theta_1$. Thus, $F(\pi) \neq F_{D,\delta}(\pi)$ for any fair partition of N, and therefore the strong paradox holds.

Proof of assertion (2): The proof is similar to the one above. Let F be a scoring rule with score vector $s = (s_1, s_2, s_3)$ for profiles π with $I(\pi) = 3$, where $s_3 = 0$.

Suppose first that $s_1 = s_2$. Consider again example in Table 7. Let We get $F(\pi) = \Phi$. Moreover, $F((\pi_{D_1}) = \omega - \Theta_2$ and $F((\pi_{D_2}) = \omega - \Theta_0$. Thus $\varphi((\pi_{D_1}) \cup \varphi((\pi_{D_2}) = \omega$. It follows that $F[\delta(\pi, D) \mid \cup_{t=1,2} F(\pi_{D_t})] = F_{D,\delta}(\pi) = \Phi \cup \Theta_1 \neq F(\pi)$ for all fair partitions D.

Finally, suppose that $s_1 > s_2$. Let $N = \{1, 2, 3, 4, 5\}$, and let $\omega = \Theta \cup \Phi \cup \{b\}$, where Θ (resp. Φ) is the set of one-to-one mappings from $\{1, 1', 2, 2', 2''\}$ (resp. $\{1, 1', 1'', 2, 2'\}$) to N. Preferences π are such that, for any $i \in N$, for any $x \in \omega - \{b\}$, $Rank_{\pi_i}(x) = r \in \{1, 2\}$ if and only if $x^{-1}(i) \in \{r, r', r''\}$, and $Rank_{\pi_i}(b) = 3$. Since $s_1 > s_2$, then $F(\pi) = \Phi$. Any fair partition D is such that $D = \{D_1, D_2\}$, where $D_1 = \{i, i'\}$ and $D_2 = \{j, j', j''\}$. We get a situation similar to the one given in Tables 5 and 6. Preferences over $\Theta \cup \Phi$ in each of the two districts are described in Table 8:

Table 8

$x \in$	Δ_1	Δ_2	Δ_3	Φ_1	Φ_2	Φ_3
$(Rank_{\pi_i}(x), Rank_{\pi_{i'}}(x))$	$(1,1)$	$(1,2)$	$(2,2)$	$(1,1)$	$(1,2)$	$(2,2)$
$(Rank_{\pi_j}(x), Rank_{\pi_{j'}}(x), Rank_{\pi_{j''}}(x))$	$(2,2,2)$	$(1,2,2)$	$(1,1,2)$	$(1,2,2)$	$(1,1,2)$	$(1,1,1)$
$s_{rank_{\pi_i}}(x) + s_{rank_{\pi_{i'}}}(x)$	$2s_1$	$s_1{+}s_2$	$2s_2$	$2s_1$	$s_1{+}s_2$	$2s_2$
$s_{rank_{\pi_j}}(x) + s_{rank_{\pi_{j'}}}(x) + s_{rank_{\pi_{j''}}}(x)$	$3s_2$	$s_1{+}2s_2$	$2s_1{+}s_2$	$s_1{+}2s_2$	$2s_1{+}s_2$	$3s_1$
$Rank_{\widetilde{F}(\pi_{D_1})}(x)$	1	2	3	1	2	3
$Rank_{\widetilde{F}(\pi_{D_2})}(x)$	4	3	2	3	2	1

Thus $F((\pi_{D_1}) \cup F((\pi_{D_2}) = \{\Theta_1, \Phi_1, \Phi_3\}$. Since $F(\pi) = \Phi$, then $F(\pi) \neq F[\delta(\pi, D) \mid \cup_{t=1,2} F(\pi_{D_t})]$ for any fair partition D, which completes the proof.