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Triple-consistent Social Choice and the Majority Rule

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Abstract We define generalized (preference) domains $\mathcal{D}$ as subsets of the hypercube $\{-1, 1\}^D$, where each of the D coordinates relates to a yes-no issue. Given a finite set of n individuals, a profile assigns each individual to an element of $\mathcal{D}$. We prove that the outcome of issue-wise majority voting φ_m belongs to $\mathcal{D}$ at any profile where φ_m is well-defined if and only if this is true when φ_m is applied to any profile involving only 3 elements of $\mathcal{D}$. We call this property triple-consistency. We characterize the class of anonymous issue-wise voting rules that are triple-consistent, and give several interpretations of the result, each being related to a specific collective choice problem.

Key words Majority Rule – Triple-consistency – Multiple Elections – Stable Domains – Arrowian aggregation
Mathematics Subject Classification (2000): 91B12, 91B14

1 Introduction

A Condorcet preference domain is a set of linear orders over finitely many alternatives where majority rule applies without the Condorcet effect: at any profile involving finitely many individuals, that assigns each individual some order in the set, majority relation does not produce cycles. The celebrated Condorcet paradox states that the complete domain is not Condorcet. This calls for finding restrictions upon preferences which secure acyclic majority outcomes (Inada (1969), Sen and Pattanaik (1969), see Gaertner (1979) for a survey). A large literature is devoted to identifying maximal Condorcet domains[1], (Fishburn (1997), (2002), see Montjardet (2010) for a recent survey). The well-known Black's single-peakedness property (Black (1948)) defines a well-known maximal Condorcet domain (although a small one). Moreover, majority rule provides a single-peaked linear order when applied to any profile of single-peaked linear orders subscribed by an odd number of individuals. Put differently, the domain of single-peaked preference is *stable* under majority rule.

Since majority rule satisfies unanimity and independence of irrelevant alternatives, a domain is stable under majority rule only if majority voting is an Arrovian aggregation rule[2]. A recent flow of research

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[1] A Condorcet domain is maximal if it is no longer Condorcet when completed by any additional order.

[2] An Arrovian aggregation rule is a mapping φ from a set of profiles π of linear orders over a set $\mathcal{A}$ of alternatives to the set of all linear orders over $\mathcal{A}$ which satisfies:

(see Nehring and Puppe (2010), Dokow and Holzman (2010), and the references quoted there) extends the study of Condorcet domains to a more general setting. Define a generalized domain as a set $\mathcal{D}$ of (non-necessarily transitive) asymmetric and complete binary relations over a finite set of alternatives. Define an aggregator as a mapping from $\mathcal{D}^n$ to $\mathcal{D}$, where n is a variable number of individuals, each being endowed with an element of $\mathcal{D}$. How to characterize domains where an Arrovian aggregator exists?

Following Wilson (1975), a generalized domain can be formalized as a subset $\mathcal{D}$ of the hypercube $\{-1,1\}^D$, where each coordinate $d = 1,...,D$ is called an issue. In the classical case of preference aggregation, an issue d is a pair $\{a,b\}$ of alternatives, and elements $x = (x^1,...,x^D)$ of $\mathcal{D}$ are defined by $x^d = 1$ (resp. -1) if a is ranked above (resp. below) b. Hence, if $D = \frac{m(m-1)}{2}$, any set of asymmetric and complete binary relations over m alternatives can be defined as a subset of $\{-1,1\}^D$. In case of linear orders, transitivity clearly imposes additional constraints between coordinates. More generally, since this setting allows for any logical restriction between coordinates, it fits with a wide range of collective decision-making problems[3].

Given a generalized domain $\mathcal{D}$ together with a set of n individuals, a profile is an element of $\mathcal{D}^n$. An aggregation rule φ maps each profile to an element of $\{-1,1\}^D$. Hence, $\mathcal{D}$ is stable under φ if and only if φ is an aggregator for any n. A characterization of stable domains under majority rule is given by Nehring and Puppe (2010).

We follow in this paper a dual approach, which focuses on a computational property of aggregation rules, rather than on the structure of stable domains. This property, called *triple-consistency,* is defined for all possible domains and for a variable number of individuals: an aggregation rule φ is triple-consistent when for any domain $\mathcal{D} \subseteq \{-1,1\}^D$ is stable under φ if and only if applying φ to any triple of not all identical elements of $\mathcal{D}$ gives an element of $\mathcal{D}$. Triple-consistency provides a simple test for the stability of any generalized domain.

Our starting point is to observe that majority rule is triple-consistent (Theorem 2). Then we investigate whether triple-consistency holds for other aggregation rules. Our main result (Theorem 3) is that an anonymous and independent aggregation rule is triple consistent if and only if either it is the majority rule, or an almost constant rule, or an alternate majority rule. An aggregation rule is almost constant if it always selects for each issue d the same position regardless the preference profile, unless this profile is unanimous about d. Furthermore, an aggregation rule is an alternate majority rule if it coincides with the anti-majority rule for any profile with 3 individuals, and either with the majority rule or the anti-majority rule for profiles with an higher (odd) number of individuals. As an immediate consequence of Theorem 3, one gets that the issue-wise majority rule is the unique anonymous, neutral, independent and unanimous issue-wise aggregation rule which is triple-consistent.

The paper is organized as follows. Aggregation rules for generalized domains are formalized in Part 2. Results are presented in Part 3. Finally, we discuss in Part 4 interpretations and consequences of the results for alternative collective decision problems.

2 Abstract Arrovian Aggregation

A domain is a finite subset $\mathcal{D}$ of $\{-1,1\}^D$, where coordinates $d = 1,...,D$ are called issues and elements $x = (x^1,...,x^D)$ of $\mathcal{D}$ are called programs. We say that x approves d (resp. disapproves d) if $x^d = 1$ (resp. -1). The program opposite to x is denoted by $(-x) = (-x^1,...,-x^D)$, and we define $(-\mathcal{D}) = \{x \in \{-1,1\}^D : (-x) \in \mathcal{D}\}$. We denote by Δ the set of all possible domains.

We provide below a formal model of social choice for any domain $\mathcal{D}$ and a variable number n of individuals.

$\mathbb{N}$ stands for the set of non-zero natural numbers. Given a domain $\mathcal{D}$ together with a number $n \geq 3$ of individuals, a $(\mathcal{D},n)$-profile is an element $\pi_n^{\mathcal{D}} = (x_1,...,x_n)$ of $\mathcal{D}^n$, where $x_i = (x_i^1,...,x_i^D)$ describes the

- If all preferences in π rank $a \in \mathcal{A}$ above $a' \in \mathcal{A}$, the so does $\varphi(\pi)$ (unanimity).
- If two profiles π and π' coincide in restriction to a pair $\{a,a'\}$ of alternatives, then $\varphi(\pi)$ and $\varphi(\pi')$ rank a and a' in the same way (independence of irrelevant alternatives).

[3] In particular, it relates to the problem of judgment aggregation, which has generated a vaste literature (see List and Puppe (2009) for a survey). We provide below other interpretations of generalized domains.

position of individual $i \in \{1,...,n\}$ about each of the issues. Given issue d, we define $\pi_n^{\mathcal{D}} \mid_d = (x_1^d,...,x_n^d) \in \{-1,1\}^n$ as the vector of all individual positions regarding d. Furthermore, $n_1(\pi_n^{\mathcal{D}} \mid_d)$ (resp. $n_{-1}(\pi_n^{\mathcal{D}} \mid_d)$) denotes the number of individuals having position 1 (resp. -1) on d in profile $\pi_n^{\mathcal{D}}$. A $(\mathcal{D},n)$-profile $\pi_n^{\mathcal{D}}$ is *non-unanimous* on d if $n_1(\pi_n^{\mathcal{D}} \mid_d) \neq n$ and $n_{-1}(\pi_n^{\mathcal{D}} \mid_d) \neq n$. Moreover, $\pi_n^{\mathcal{D}}$ is non-unanimous if it is non-unanimous on some issue d. The set of all possible $(\mathcal{D},n)$-profiles is denoted by $\Pi_n^{\mathcal{D}}$.

An *aggregation rule* is a function φ from $\cup_{\mathcal{D}\in\Delta}\cup_{n\geq 3}\Gamma_n^{\mathcal{D}}$ to $\cup_{D\in\mathbb{N}}\{-1,1\}^D$ such that $\forall\mathcal{D}\in\Delta$, $\forall n\geq 3$, $\Gamma_n^{\mathcal{D}} \subseteq \Pi_n^{\mathcal{D}}$ and $\forall \pi_n^{\mathcal{D}} \in \Gamma_n^{\mathcal{D}}$, $\varphi(\pi_n^{\mathcal{D}}) \in \{-1,1\}^D$. $\Gamma_n^{\mathcal{D}}$ is defined as the largest set of $(\mathcal{D},n)$-profiles $\pi_n^{\mathcal{D}}$ for which $\varphi(\pi_n^{\mathcal{D}})$ is well-defined. Elements of $\Gamma_n^{\mathcal{D}}$ are called admissible $(\mathcal{D},n)$-profiles for φ. Hence, an aggregation rule maps each admissible $(\mathcal{D},n)$-profile to a program over D issues. We denote by $[\varphi(\pi_n^{\mathcal{D}})]^d$ the position in $\varphi(\pi_n^{\mathcal{D}})$ about issue d. A well-known aggregation rule is the *issue-wise majority rule* φ_m, which selects issue–wise the position that gathers the largest support among individuals. Clearly, $\Gamma_n^{\mathcal{D}}$ is for φ_m the subset of $\Pi_n^{\mathcal{D}}$ containing all profiles with an odd number of individuals. Formally, for all $\mathcal{D}\in\Delta$, for all $n\in 2\mathbb{N}+1$, for all $\pi_n^{\mathcal{D}} = (x_1,...,x_n) \in \Pi_n^{\mathcal{D}}$, and for all $d\in\{1,...,D\}$, $[\varphi_m(\pi_n^{\mathcal{D}})]^d = 1$ if and only if $n_1(\pi_n^{\mathcal{D}} \mid_d) > n_{-1}(\pi_n^{\mathcal{D}} \mid_d)$.

We introduce below several properties for aggregation rules.

- φ is *anonymous* if for all $\mathcal{D}\in\Delta$ and all $n\geq 3$, for all $\pi_n^{\mathcal{D}} = (x_1,...,x_n) \in \Gamma_n^{\mathcal{D}}$, and for any permutation σ of $\{1,...,n\}$, we have $\varphi(\pi_n^{\mathcal{D}}) = \varphi(\sigma(\pi_n^{\mathcal{D}}))$, where $\sigma(\pi_n^{\mathcal{D}}) = (x_{\sigma(1)},...,x_{\sigma(n)})$. Anonymity means that the outcome of φ is non-sensitive to the names of individuals.

- φ is *independent* if for all $\mathcal{D}\in\Delta$ and all $n\geq 3$, for any two $(\mathcal{D},n)$-profiles $\pi_n^{\mathcal{D}}$ and $\widetilde{\pi}_n^{D}$ in $\Gamma_n^{\mathcal{D}}$, and for any two issues $d,d'\in\{1,...D\}$ such that $\pi_n^{\mathcal{D}} \mid_d = \widetilde{\pi}_n^{D} \mid_{d'}$, we have $[\varphi(\pi_n^{\mathcal{D}})]^d = [\varphi(\widetilde{\pi}_n^{D})]^{d'}$. Independence states that aggregation of programs is made issue by issue, and that it does not depend on the names of issues.[4]

- φ is *neutral* if for all $\mathcal{D}\in\Delta$ and all $n\geq 3$, for all $\pi_n^{\mathcal{D}} = (x_1,...,x_n) \in \Gamma_n^{\mathcal{D}}$ and for any issue d, we have $[\varphi(\pi_n^{\mathcal{D}})]^d = -[\varphi(\widehat{\pi}_n^{D})]^d$, where $\widehat{\pi}_n^{D} = (\widehat{x}_1,...,\widehat{x}_n)$ is the $(\mathcal{D},n)$-profile defined by: $\forall i\in\{1,...,n\}$, $\widehat{x}_i^d = -x_i^d$ and $\widehat{x}_i^{d'} = x_i^{d'}$ for all $d'\neq d$. In words, if all positions regarding some issue d are reversed while all positions on other issues do not change, the social position on d should be reversed. Neutrality means that the social outcome on any issue is non-sensitive to the labelling of positions on that issue. While independence states that the same type of aggregation prevails on each of the issues, neutrality refers to a symmetric treatment of issue-wise positions.

- φ is *unanimous* if for all $\mathcal{D}\in\Delta$ and all $d\in\{1,...,D\}$, for all $n\geq 3$, for all $a\in\{-1,1\}$ and for all $\pi_n^{\mathcal{D}} \in \Gamma_n^{\mathcal{D}}$, we have $[\varphi(\pi_n^{\mathcal{D}})]^d = a$ if $n_a(\pi_n^{\mathcal{D}} \mid_d) = n$: if all individuals share the same position on some issue, this position should be socially chosen.

The issue-wise majority rule φ_m is obviously anonymous, independent, neutral and unanimous (on all profiles with an odd number of individuals).

Other anonymous and independent aggregation rules are:

- The *anti-majority rule* φ_m^-, defined by: $\forall\mathcal{D}\in\Delta$, $\forall n\in 2\mathbb{N}+1$, $\forall\pi_n^{\mathcal{D}}\in\Pi_n^{\mathcal{D}}$, $\varphi_m^-(\pi_n^{\mathcal{D}}) = -\varphi_m(\pi_n^{\mathcal{D}})$. The anti-majority rule selects issue-wise the position of the minority. Note that φ_m^- is neutral but not unanimous.

- The *unanimity rule* φ_u, which agrees issue-wise with unanimity when it prevails, and chooses a default option (interpreted as a status quo) otherwise. Formally, for all $\mathcal{D}\in\Delta$ and for all $n\geq 3$, there exists $a\in\{-1,1\}$ such that for all $\pi_n^{\mathcal{D}}\in\Pi_n^{\mathcal{D}}$, for all $1\leq d\leq D$, $n_{-a}(\pi_n^{\mathcal{D}} \mid_d) = n \Leftrightarrow [\varphi_u(\pi_n^{\mathcal{D}})]^d = -a$. Note that φ_u is unanimous but not neutral.

It is easily seen that an aggregation rule φ is independent and anonymous if and only if at any admissible $(\mathcal{D},n)$-profile $\pi_n^{\mathcal{D}}$, we have $[\varphi(\pi_n^{\mathcal{D}})]^d = g(n_1(\pi_n^{\mathcal{D}} \mid_d))$, where g is a function from $\{1,...,n\}$ to $\{-1,1\}$: the social position regarding any issue d depends only on the number of approvals (and disapprovals) given to d. The class of independent and anonymous aggregation rules is rather vast. In

[4] In the Arrovian case of preference aggregation, where issues d are pairs of alternatives, independence implies the property of independence of irrelevant alternatives. Independence is equivalently defined as follows: $\exists f : \cup_{n\geq 3}\{-1,1\}^n \rightarrow \{-1,1\}$ such that, for all $\mathcal{D}\in\Delta$ and all $n\geq 3$, for all $\pi_n^{\mathcal{D}}\in\Gamma_n^{\mathcal{D}}$, $\varphi(\pi_n^{\mathcal{D}}) = (f(\pi_n^{\mathcal{D}} \mid_1),...,f(\pi_n^{\mathcal{D}} \mid_D))$.

particular, it allows for rules showing complex patterns of non-monotonicity[5]. For example, it contains the aggregation rule $\overline{\varphi}$ defined by $\forall \mathcal{D} \in \Delta$, $\forall n \geq 3$, $\forall \pi_n^{\mathcal{D}} \in \Pi_n^{\mathcal{D}}$, $\forall d \in \{1, ..., D\}$, $[\overline{\varphi}(\pi_n^{\mathcal{D}})]^d = 1 \Leftrightarrow n_1(\pi_n^{\mathcal{D}} \mid_d) \in 2\mathbb{N}$: an issue d is socially approved if and only if it receives an even number of individual approvals.

We focus below on the following concept of stable domain.

Definition 1 *A domain $\mathcal{D}$ is stable under aggregation rule φ if for all $n \geq 3$, $\varphi(\pi_n^{\mathcal{D}}) \in \mathcal{D}$ for any $(\mathcal{D}, n)$-profile π in $\Gamma_n^{\mathcal{D}}$.*

It is easy to find domains that are stable neither under φ_m, nor φ_m^- nor φ_u. Indeed, let $\mathcal{D} = \{x \in \{-1, 1\}^3 : x^d = 1$ for exactly 2 issues $d\}$. Let $\pi_3^3 = (x_1, x_2, x_3) \in \Pi_3^3$ with $x_1 = (1, 1, -1)$, $x_2 = (1, -1, 1)$ and $x_3 = (-1, 1, 1)$. Then $\varphi_m(\pi_3^3) = (1, 1, 1) \notin \mathcal{D}$, $\varphi_m^-(\pi_3^3) = (-1, -1, -1) \notin \mathcal{D}$ and, since π_3^3 is unanimous on no issue, $\varphi_u(\pi_3^3) \in \{\varphi_m(\pi_3^3), \varphi_m^-(\pi_3^3)\}$.[6] For each issue $d \in \{1, 2, 3\}$, define $H^d = \{x \in \mathcal{D} : x^d = 1\}$ and $H_C^d = \{x \in \mathcal{D} : x^d = -1\}$. Then the family of subsets $\mathcal{H} = \{H^1, H^2, H^3\}$ is such that $\cap_{d=1}^3 H^d = \varnothing$, while $\cap_{d \neq d'} H^d \neq \varnothing$ for all $d' \in \{1, 2, 3\}$. Nehring and Puppe (2010) call $\mathcal{H}$ a critical family, and characterize stable domains under φ_m through the cardinality of critical families. Translating their approach into our terminology, Nehring and Puppe define a domain $\mathcal{D}$ as a subset of $\{-1, 1\}^D$ such that $\forall d$, $H^d \neq \varnothing$ and $H_C^d \neq \varnothing$, where H^d is called property for d, and H_C^d complementary property for d. [7] A family $\mathcal{H}$ of properties is a subset of $[\cup_{d=1}^D H^d] \cup [\cup_{d=1}^D H_C^d]$. Moreover, $\mathcal{H}$ is critical for $\mathcal{D}$ if (1) $\cap \mathcal{H} = \varnothing$ and (2) $\forall H \in \mathcal{H}$, $\cap\{\mathcal{H} - \{H\}\} \neq \varnothing$.

Theorem 1 *(Nehring and Puppe 2010) A domain $\mathcal{D}$ is stable under φ_m if and only if all families of properties that are critical for $\mathcal{D}$ have cardinality 2.*

For an illustration, define $\mathcal{D} = \{(1, -1, 1, 1), (-1, -1, 1, -1), (1, 1, 1, -1), (1, -1, -1, -1)\}$. Then we note that $\mathcal{H} = \{H^1, H_C^2, H^3, H_C^4\}$ is critical. Indeed, $\mathcal{D}$ does not contain program $(1, -1, 1, -1)$, but contains programs of type $(a, -1, 1, -1)$, $(1, a, 1, -1)$, $(1, -1, a, -1)$ and $(1, -1, 1, a)$, where $a \in \{-1, 1\}$. Thus, since $\mathcal{H}$ has more than 2 elements, there exists a number n of individuals together with a profile π_n^4 in $\mathcal{D}^n$ such that $\varphi_m(\pi_n^4) \notin \mathcal{D}$.

We focus here on aggregation rules rather than on domain structure, and we provide a computationally simple test for domain stability under majority rule. This test does not relate to subsets of issues (or properties) to be checked, but instead relates to the size of profiles to be considered. Moreover, we characterize the class of all independent and anonymous aggregation rules for which the test applies.

3 Results

We introduce in the next definition the notion of 3-stable domain:

Definition 2 *A domain $\mathcal{D}$ is 3-stable under aggregation rule φ if $\varphi(x, y, z) \in \mathcal{D}$ for any 3-tuple of programs (x, y, z) in $\mathcal{D}^3$ admissible for φ.*

Obviously, 3-stability is necessary to stability. Maybe less intuitive is that it also sufficient for stability under φ_m.

Theorem 2 *A domain $\mathcal{D}$ is stable under φ_m if and only if $\mathcal{D}$ is 3-stable under φ_m.*

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[5] An anonymous and independent aggregation rule φ is monotonic if $\forall \mathcal{D} \in \Delta$, $\forall n \geq 3$, $\forall \pi_n^{\mathcal{D}} \in \Gamma_n^{\mathcal{D}}$, $\forall d \in \{1, ..., D\}$, $\forall a \in \{-1, 1\}$, $[\varphi(\pi_n^{\mathcal{D}})]^d = a$ only if $[\varphi(\widetilde{\pi}_n^{\mathcal{D}})]^d = a$ for any $\widetilde{\pi}_n^{\mathcal{D}} \in \Gamma_n^{\mathcal{D}}$ such that $n_a(\widetilde{\pi}_n^{\mathcal{D}} \mid_d) > n_a(\pi_n^{\mathcal{D}} \mid_d)$.

[6] A possible interpretation of this example is that a program is a list of potential public expenses in a society facing a budget constraint that allow to undertake exactly two expenses. Individuals claim for different expenses under the commonly known budget constraint. By using any of the three rules φ_m, φ_m^- and φ_u, the society may fail at meeting the constraint.

[7] Note that we do not retain this restriction in our definition of a domain.

Proof Suppose that $\mathcal{D}$ is 3-stable under φ_m. Suppose for a contradiction that there exists $n \in 2\mathbb{N}+1$ and $\pi_n^{\mathcal{D}} = (x_1, ..., x_n) \in \Pi_n^{\mathcal{D}}$ such that $\varphi_m(\pi_n^{\mathcal{D}}) \notin \mathcal{D}$. In order to lighten notations, we write $\pi_n^{\mathcal{D}} = \pi$. Let x_0 be the program with the highest number of issues, denoted by k, such that $x_0^d = [\varphi_m(\pi)]^d$. Note that issues can be re-ordered so that $x_0^d = [\varphi_m(\pi)]^d \Leftrightarrow d = 1, ..., k$. Define $I = \{i \in \{1, ..., n\} : x_i^{k+1} = [\varphi_m(\pi)]^{k+1}\}$. The definition of φ_m implies that $\forall d$, $\exists i(d) \in I$ such that $x_{i(d)}^d = [\varphi_m(\pi)]^d$. Indeed, we necessarily have $|I| > \frac{n}{2}$, and if there exists d with $x_i^d \neq [\varphi_m(\pi)]^d$ for all $i \in I$, then the majority position on d is $-[\varphi_m(\pi)]^d$, clearly a contradiction. Let $z_2 = \varphi_m(x_0, x_{i(1)}, x_{i(2)})$. By 3-stability, $z_2 \in \mathcal{D}$, and $z_2^1 = [\varphi_m(\pi)]^1$, $z_2^2 = [\varphi_m(\pi)]^2$ and $z_2^{k+1} = [\varphi_m(\pi)]^{k+1}$. Then, for any $3 \leq h \leq k$, define z_h by $z_h = \varphi_m(x_0, z_{h-1}, x_{i(h)})$. Using again 3-stability, one get that $z_h \in \mathcal{D}$, $z_h^d = [\varphi_m(\pi)]^d$ for all $1 \leq d \leq h$ and $z_h^{k+1} = [\varphi_m(\pi)]^d$. Thus, z_k agrees with $\varphi_m(\pi)$ on at least $k+1$ issues, and this contradicts the definition of x_0 ♦

Hence, given any domain $\mathcal{D}$, suppose that, when picking up any 3-tuple (x, y, z) of programs in $\mathcal{D}$, the outcome of $\varphi_m(x, y, z)$ is a program in $\mathcal{D}$. Then, for all $n \in 2\mathbb{N}+1$, $\varphi_m(\pi_n^{\mathcal{D}})$ will also be a program in $\mathcal{D}$ for any (D, n)-profile $\pi_n^{\mathcal{D}}$. This property is formalized in the next definition.

Definition 3 *An aggregation rule φ is triple-consistent if for all domains $\mathcal{D} \in \Delta$, $\mathcal{D}$ is stable under φ if and only if $\mathcal{D}$ is 3-stable under φ.*

We now address is the following question: can we find independent and anonymous triple-consistent aggregation rules beyond φ_m? The answer is trivially positive. Indeed, define the constant rule φ_{cons} by: $\forall \mathcal{D} \in \Delta$, there exists $x_{\mathcal{D}} \in \{-1, 1\}^D$ such that $\forall n \in \mathbb{N}$, $\forall \pi_n^{\mathcal{D}} \in \Pi_n^{\mathcal{D}}$, $\varphi_{cons}(\pi_n^{\mathcal{D}}) = x_{\mathcal{D}}$. Since any domain $\mathcal{D}$ that is 3-stable under φ_{cons} must contain $x_{\mathcal{D}}$, then $\mathcal{D}$ is stable under φ_{cons}. Triple-consistency also holds true for the larger class of almost constant aggregation rules. We say that an aggregation rule φ is *almost constant* if $\forall \mathcal{D} \in \Delta$, $\forall 1 \leq d \leq D$, $\forall n \geq 3$, $\forall \pi_n^{\mathcal{D}}, \widetilde{\pi}_n^{\mathcal{D}} \in \Pi_n^{\mathcal{D}}$ non-unanimous on d, we have $[\varphi(\pi_n^{\mathcal{D}})]^d = [\varphi(\widetilde{\pi}_n^{\mathcal{D}})]^d$.[8]

Proposition 1 *Every almost constant independent aggregation rule is triple-consistent.*

Proof Let φ be independent and almost constant. From definition, $\forall \mathcal{D} \in \Delta$, $\forall 1 \leq d \leq D$, $\forall \pi_n^{\mathcal{D}} \in \Pi_n^{\mathcal{D}}$ non-unanimous on d, $[\varphi(\pi_n^{\mathcal{D}})]^d = a_d \in \{-1, 1\}$. Moreover, if $\pi_n^{\mathcal{D}}$ is unanimous on d with position $b_d \in \{-1, 1\}$, then either $[\varphi(\pi_n^{\mathcal{D}})]^d = b_d$ (if φ is unanimous) or $[\varphi(\pi_n^{\mathcal{D}})]^d = -b_d$ (φ is not unanimous). Let $\mathcal{D}$ be 3-stable under φ, and let $\pi_n^{\mathcal{D}} = (x_1, x_2, ..., x_n) \in \Pi_n^{\mathcal{D}}$, with $y = \varphi(\pi)$. Furthermore, define $z_3 = \varphi(x_1, x_2, x_3)$, and, for $4 \leq k \leq n$, $z_k = \varphi(x_1, z_{k-1}, x_k)$. From 3-stability, $z_k \in \mathcal{D}$ for all $3 \leq k \leq n$. We claim that $z_n = y$. Indeed, suppose that $y^d = -a_d$. Thus, $\pi_n^{\mathcal{D}}$ is unanimous on d. If φ is unanimous (resp. not unanimous), then $x_i^d = -a_d$ (resp. $x_i^d = a_d$) for all $1 \leq i \leq n$, so that $z_k^d = -a_d$ for all $k \geq 3$, hence $z_k^d = y^d$. Finally, suppose that $y^d = a_d$. If $\pi_n^{\mathcal{D}}$ is unanimous on d, we get again that $z_k^d = a_d$ for all $k \geq 3$. If $\pi_n^{\mathcal{D}}$ is not unanimous on d, then $\exists k_d$ such that $x_{k_d}^d \neq x_1^d$, and hence one get $z_k^d = a_d$ for all $k \geq k_d$ ♦

Proposition 1 implies in particular that φ_u is triple-consistent. Another class of triple-consistent and independent rules is the class of *alternate majority rules* Φ_m^{+-}, defined as follows: $\varphi \in \Phi_m^{+-}$ if for all $D \in \mathbb{N}$, for all $n \in 2\mathbb{N}+1$, for all $\pi_n^{\mathcal{D}} \in \Pi_n^{\mathcal{D}}$, $\varphi(\pi_n^{\mathcal{D}}) \in \{\varphi_m(\pi_n^{\mathcal{D}}), \varphi_m^-(\pi_n^{\mathcal{D}})\}$ if $n > 3$ and $\varphi(\pi_3^{\mathcal{D}}) = \varphi^-(\pi_3^{\mathcal{D}})$. Hence φ^{+-} selects a program according to the minority will in the 3-individual case, and either according to the majority or the minority will in all other cases.

Proposition 2 *Every alternate majority rule is triple-consistent.*

Proof Let $\varphi \in \Phi_m^{+-}$. Note first that, given any domain $\mathcal{D}$ together with $x, y \in \mathcal{D}$, $\varphi(x, x, y) = -x$. Indeed, for all d, if $x^d \neq y^d \Rightarrow \varphi(x, x, y) = y^d = -x^d$, and $x^d = y^d \Rightarrow \varphi(x, x, y) = -y^d = -x^d$. Thus, if $\mathcal{D}$ is 3-stable under φ, one must have $\forall x \in \mathcal{D}$, $-x \in \mathcal{D}$. This implies that $\mathcal{D}$ is 3-stable under φ_m^- only if $\mathcal{D}$ is 3-stable under φ_m. Consider a domain $\mathcal{D}$ which is 3-stable under φ. Pick up $n \in 2\mathbb{N}+1$ and any

[8] Note that if φ is independent, then φ is almost constant if there exists $a \in \{-1, 1\}$ such that for all $\mathcal{D} \in \Delta$, for all $n \geq 3$, and for all $\pi_n^{\mathcal{D}} \in \Pi_n^{\mathcal{D}}$, we have $0 < n_{-a}(\pi_n^{\mathcal{D}}) < n \Rightarrow [\varphi(\pi_n^{\mathcal{D}})]^d = a$.

$(\mathcal{D}, n)$-profile $\pi_n^{\mathcal{D}}$. From Theorem 1, $\varphi_m(\pi_n^{\mathcal{D}}) \in \mathcal{D}$, so that $\varphi(\pi_n^{\mathcal{D}}) = \varphi_m(\pi_n^{\mathcal{D}}) \Rightarrow \varphi(\pi) \in \mathcal{D}$. Finally, if $\varphi(\pi_n^{\mathcal{D}}) = \varphi_m^-(\pi_n^{\mathcal{D}})$, then $\varphi(\pi) = -\varphi_m(\pi)$. By 3-stability, $\varphi_m(\pi_n^{\mathcal{D}}) \in \mathcal{D} \Rightarrow -\varphi_m(\pi_n^{\mathcal{D}}) \in \mathcal{D}$, and the proof is complete ⧫

Theorem 3 below (proved in the Appendix) characterizes the class of anonymous, independent and triple-consistent aggregation rules:

Theorem 3 *An aggregation rule φ is independent, anonymous, and triple-consistent if and only if either $\varphi = \varphi_m$, or $\varphi \in \Phi_m^{+-}$, or φ is almost constant.*

Since no almost constant rule is neutral, and since no alternate majority rule is unanimous, we get as an immediate consequence of Theorem 3:

Corollary 1 *(1) An aggregation rule φ is independent, anonymous, neutral and triple-consistent if and only if $\varphi \in \{\varphi_m\} \cup \Phi_m^{+-}$.*

(2) An aggregation rule φ is independent, anonymous, unanimous, and triple-consistent if and only if $\varphi \in \{\varphi_m, \varphi_u\}$.

(3) An aggregation rule φ is independent, anonymous, unanimous, neutral, and triple-consistent if and only if $\varphi = \varphi_m$.

We conclude this Part by noting that φ_m is no longer triple-consistent when completed by a tie-breaking rule for profiles with an even number of individuals:

Proposition 3 *Let φ be an independent and anonymous aggregation rule such that $\varphi(\pi_n^{\mathcal{D}}) = \varphi_m(\pi_n^{\mathcal{D}})$ for all $(\mathcal{D}, n)$-profiles $\pi_n^{\mathcal{D}}$ with n odd, and for $(\mathcal{D}, n)$-profiles $\pi_n^{\mathcal{D}}$ with n even, there exists $a \in \{-1, 1\}$ such that, for some issue d, $n_a(\pi_n^{\mathcal{D}}\,|_d) = n_{-a}(\pi_n^{\mathcal{D}}\,|_d) \Rightarrow [\varphi(\pi_n^{\mathcal{D}})]^d = a$. Then φ is not triple-consistent.*

Proof Suppose that $a = 1$. Let $\mathcal{D} = \{x, y, z, t\} \subset \{-1, 1\}^3$, with $x = (1, -1, 1)$, $y = (-1, 1, -1)$, $z = (-1, -1, 1)$, and $t = (1, 1, -1)$. Then $\varphi(x, y, z) = z$, $\varphi(x, y, t) = t$, $\varphi(x, z, t) = x$ and $\varphi(y, z, t) = y$. Moreover, the definition of φ ensures that $\forall w \neq w' \in \mathcal{D}$, $\varphi(w, w, w') = w \in \mathcal{D}$. Hence, $\mathcal{D}$ is 3-stable under φ. Finally, consider the $(\mathcal{D}, 2h+2h')$-profile $\pi_{2h+2h'}^4 = (\overbrace{x, ..., x}^{h}, \overbrace{y, ..., y}^{h}, \overbrace{z, ..., z}^{h'}, \overbrace{t, ..., t}^{h'})$. Then $\varphi(\pi_{2h+2h'}^4) = (-1, 1, 1, 1) \notin \mathcal{D}$. Hence $\mathcal{D}$ is not stable under φ. Therefore φ is not triple-consistent. The case where $a = -1$ is solved by replacing $\mathcal{D}$ by $(-\mathcal{D}) = \{-x, -y, -z, -t\}$ ⧫

4 Discussion

Our results depart from the classical approach of domain restriction by focusing on a computational property of aggregation rules that holds *for any (generalized) domain.* A first implication relates to the Arrovian framework of preference aggregation. Defining a domain $\mathcal{D}$ as a set of linear orders over a finite number m of alternatives (implying $\frac{m(m-1)}{2}$ issues), Corollary 1 (2) states that an anonymous Arrovian aggregation rule is triple-consistent if and only if it is either the majority rule or the unanimity rule. Furthermore, a well-known result is that any Condorcet domain $\mathcal{D}$ of linear orders is characterized by the following value-restriction property (Sen (1966)): for every subset $\{a, b, c\}$ of alternatives, there exists $e \in \{a, b, c\}$ which either never has rank 1 or never has rank 2 or never has rank 3 in the restriction $\mathcal{D}/\{a, b, c\}$. Maskin (1995) remarks that the value-restriction property works regardless the number of individuals in a profile drawn from $\mathcal{D}$. Thus, majority rule provides a transitive preference for any odd $\mathcal{D}$-profile if and only if it does so with three individual preferences drawn from $\mathcal{D}$. The idea of triple-consistency originates from this remark, which also, together with Theorem 2, implies the following

Proposition 4 *For any Condorcet domain $\mathcal{D}$, there exists a Condorcet domain $\mathcal{D}'$ that is stable under φ_m and contains $\mathcal{D}$.*

Proof Since $\mathcal{D}$ is Condorcet, then for any profile $\pi^{(1)} = (x_1, x_2, x_3) \in \mathcal{D}^3$ with exactly 3 individuals and any triple of alternatives $\{a, b, c\}$, $\varphi_m(\pi^{(1)})$ is transitive over $\{a, b, c\}$. Denote by $\pi^{(1)} |_{\{a,b,c\}}$ the restriction of $\pi^{(1)}$ to $\{a, b, c\}$, and by $x |_{\{a,b,c\}}$ the restriction of $x \in \mathcal{D}$ to $\{a, b, c\}$. It is easily checked that $\varphi_m(\pi^{(1)} |_{\{a,b,c\}}) = x_i |_{\{a,b,c\}}$ for some $i \in \{1, 2, 3\}$. It follows from the value-restriction property that $\mathcal{D} \cup \{y_1\}$ is a Condorcet domain, with $y_1 = \varphi_m(\pi^{(1)})$. Then, pick up any other triple $\{a', b', c'\}$ and build $\pi^{(2)} = (y_1, x_4, x_5) \in [\mathcal{D} \cup \{y_1\}]^3$. Again, $\varphi_m(\pi^2 |_{\{a',b',c'\}}) \in \{y_1 |_{\{a',b',c'\}}, x_4 |_{\{a',b',c'\}}, x_5 |_{\{a',b',c'\}}\}$, and thus value-restriction ensures that $\mathcal{D} \cup \{y_1, y_2\}$ is Condorcet. When iterating the construction, it follows from the finiteness of the full domain that the process stops at some stage T, at which $\mathcal{D}' = \mathcal{D} \cup \{y_1, ..., y_T\}$ is 3-stable under φ_m. Theorem 2 allows to conclude ⧫

Another application of our results deals with the problem of judgment aggregation (List and Pettit (2002), Dietrich (2006), List and Puppe (2009)). There exists D either true or false logical propositions that are inter-related, so that only a subset of the 2^D possible combinations are logically consistent. Assigning each proposition either 1 or -1 according to truth or falsehood allows to define any domain $\mathcal{D} \subset \{-1, 1\}^D$ as a set of logically mutually consistent propositions. For instance, $\mathcal{D} = \{(-1, -1, -1), (-1, 1, -1), (1, -1, -1), (1, 1, 1)\}$ describes the case where the third proposal is true if and only if the first two are. The discursive dilemma (see Kornhauser and Sager (1986), (2004)) expresses the possible logical inconsistency of proposition-wise majority judgment. Hence, Theorem 3 provides a simple test for checking the exposure of any set of logically consistent judgments to the discursive dilemma.

Another interpretation of generalized domains is given by situations of multiple elections. Consider a society having to decide on several dichotomous, or yes-or-no issues, such as in multiple referendum, or in multi-seat elections with two competing parties (where, say, a president, a governor and a mayor are simultaneously elected). In such situations, issue-wise majority voting may result in a socially undesirable outcome. Each meaning of 'undesirable' relates to a specific 'voting paradox'[9]. The *paradox of multiple elections* (Brams, Kilgour and Zwicker (1998)) occurs when the outcome of issue-wise majority voting receives the fewest votes among all possible outcomes. A strong version of the paradox holds when the outcome is cast by no voter (Scarsini (1998)). Formally, suppose that two parties A and B compete on D positions. A domain is interpreted as an electorate, that is a set $\mathcal{B}$ of ballots $x = (x^1, ..., x^D) \in \{A, B\}^D$, where $x^d = A$ means that party A is the preferred one regarding position d. Define a vote profile as a mapping π from $\mathcal{B}$ to the set of natural numbers, where $\pi(x)$ is the number of voters having cast the ballot x. The support of π is the set $Supp(\pi) = \{x \in \mathcal{B} : \pi(x) > 0\}$. Each ballot is either actually or potentially cast by one or several voters. $\mathcal{B}$ is a *potential electorate if* $Supp(\pi) \subset \mathcal{B}$, and $\mathcal{B}$ is an *actual electorate* if $Supp(\pi) = \mathcal{B}$. Given a vote profile π, issue-wise majority rule φ_m allocates each position d to the party with the highest number of votes on d. The strong paradox of multiple elections holds at π if $\varphi_m(\pi) \notin Supp(\pi)$. An electorate is paradox-free if the strong paradox of multiple elections hold at no voting profile. We show below that checking for the paradox of multiple elections only requires looking at triples of ballots:

Proposition 5 *(1) An actual electorate is paradox-free if and only if it is 3-stable.*

(2) A potential electorate $\mathcal{B}$ is paradox-free if and only if any $\mathcal{B}' \subseteq \mathcal{B}$ containing at most 3 not all identical ballots is paradox-free.

Proof (1) Suppose $\mathcal{B}$ is 3-stable under φ_m. Hence, $\varphi_m(x, y, z) \in \mathcal{B}$ for any 3-tuple $(x, y, z) \in \mathcal{B}^3$. From Theorem 2, $\varphi_m(\pi) \in \mathcal{B}$ for any vote profile π. Thus, $\varphi_m(\pi) \in Supp(\pi)$ for any π such that $Supp(\pi) = \mathcal{B}$, and thus $\mathcal{B}$ is paradox-free. Conversely, take any 3-tuple of ballots $(x, y, z) \in \mathcal{B}^3$, and suppose that $\varphi_m(x, y, z) \notin \mathcal{B}$. Define π by $\pi(x) = \pi(y) = \pi(z) = H$, where $H \geq 1$, and $\pi(w) = 1$ for all

[9] In particular, every anonymous issue-wise voting rule may select a Pareto-dominated outcome as long as any separable preference on possible outcomes is admissible (Özkal-Sanver and Sanver (2006), Benoit and Kornhauser (2010), Cuhadaroglu and Lainé (2012)). Furthermore, under additively separable preferences over outcomes, the outcome of issue-wise majority voting may be less preferred by a majority of voters than another one (see Bezembinder and Van Acker (1985), Deb and Kelsey (1987), and Laffond and Lainé (2006), (2009)), or may disagree with the majority will on a majority of issues (Anscombe (1976), Laffond and Lainé (2013)).

$w \in \mathcal{B} - \{x, y, z\}$. If H is chosen large enough, then $\varphi_m(\pi) = \varphi_m(x, y, z) \notin \mathcal{B}$. Since $Supp(\pi) = \mathcal{B}$, then $\varphi_m(\pi) \notin Supp(\pi)$, and thus $\mathcal{B}$ faces the paradox.

(2) The necessary part is obvious. Let $\mathcal{B}$ be such that $\forall \pi$ with $Supp(\pi) \subseteq \{x, y, z\} \subseteq \mathcal{B}$, $\varphi_m(\pi) \in Supp(\pi)$. Let $x_0 \in Supp(\pi)$ that contains the largest number of issues d with $[\varphi_m(\pi)]^d = x_0^d$, and assume w.l.o.g. that $x_0^d = [\varphi_m(\pi)]^d \Leftrightarrow d = 1, ..., k < D$ (if $k = D$, then all is done since $\varphi_m(\pi) \in Supp(\pi)$). From definition of φ_m, there exists $w \neq t \in Supp(\pi)$ such that $w^{k+1} = [\varphi_m(\pi)]^{k+1}$ and $t^{k+1} = [\varphi_m(\pi)]^{k+1}$. By assumption, $\varphi_m(x_0, w, t) \in \{x_0, w, t\} \subset Supp(\pi)$. From the definition of φ_m, one gets that $[\varphi_m(x_0, w, t)]^d = [\varphi_m(\pi)]^d$ for all $d \in \{1, ..., k+1\}$, which contradicts the definition of x_0 ♦

Finally, stable generalized domains can be illustrated by the dynamics of club membership. Consider a club whose members have to decide about the appointment of new members. Conditions for membership are described through a set of D criteria, and members and candidates are characterized by the list of criteria she fulfills. Individual i's type is denoted by $x_i \in \{-1, 1\}^D$, where $x_i^d = 1$ (resp. -1) means that criterion d is (resp. not) fulfilled. Hence the current set of membership types is a generalized domain $\mathcal{D}$. Moreover, the current assembly of members is a $\mathcal{D}$-profile. Choosing new members is handled by an admission jury, defined as a sub-profile $\widetilde{\pi}$ of π. Under the assumption that every jury member promotes her own type, appointment decisions are made by means of a membership rule, defined as a mapping φ from the set of all $\mathcal{D}$-profiles to $\{-1, 1\}^D$. Hence, the jury will appoint an individual if her type coincides with $\varphi(\widetilde{\pi})$, leading to the new set of membership types $\mathcal{D}^1 = \mathcal{D} \cup \varphi(\widetilde{\pi})$. Successive admissions of new members generates a dynamics of types, since voters vote for future voters. The club is stable if $\mathcal{D}^1 = \mathcal{D}$, that is the diversity of types does not change[10]. If the membership rule is triple-consistent, the club will be stable for any admission jury if and only if it is stable for any 3-member jury. Furthermore, if anonymity and unanimity are imposed, majority rule and unanimity rule are the only criterion-wise admission rules that are triple-consistent.

5 References

Anscombe G.E.M. (1976), On the frustration of the majority by fulfillment of the majority's will, *Analysis 36:161-168*

Barberà S., Maschler M., Shalev J. (2001), Voting for voters: a model of electoral evolution, *Games and Economic Behavior 37:40-78*

Benoît J.P., Kornhauser L.A. (2010), Only dictatorship is efficient or neutral, *Games and Economic Behavior*
70:261–270

Bezembinder T., Van Acker P. (1985), The Ostrogorski paradox and its relation to non-transitive choice, *Journal of Mathematical Psychology 11:131-158*

Black D. (1948), On the rationale of group decision-making, *Journal of Political Economy 56:23–34*

Brams S., Kilgour D., Zwicker W. (1998), The paradox of multiple elections, *Social Choice and Welfare 15:211–236*

Cuhadaroglu T., Lainé J. (2012), Pareto efficiency in multiple referendum *Theory and Decision, 72:525-536*

Deb R., Kelsey D. (1987), On constructing a generalized Ostrogorski paradox: necessary and sufficient conditions, *Mathematical Social Sciences 14:161-174*

Dietrich F. (2006), Judgment Aggregation: (Im)Possibility Theorems, *Journal of Economic Theory 126:286-298*

Dokow E., Holzman R., (2010), Aggregation of binary evaluations, *Journal of Economic Theory 145:495-511*

Fishburn P.C. (1997), Acyclic sets of linear orders. *Social Choice and Welfare 14, 113–124*

[10] However, type populations in the stable club may change over time since admission juries are formed from the set of current members. Hence, stability does not means constant distribution of types. See Barbera, Mashler and Shalev (2001), and Granot, Maschler and Shalev (2002) on a related problematic.

Fishburn P.C. (2002), Acyclic sets of linear orders: A progress report. *Social Choice and Welfare 19:431–447*

Gaertner W. (1979), An analysis and comparison of several necessary and sufficient conditions for transitivity under the majority rule, *in J.J. Laffont (ed), Aggregation and revelation of preferences, North-Holland*

Granot D., Maschler M., Shalev J. (2002), Voting for voters: the unanimity case, *International Journal of Game Theory 31:55-202*

Inada K.I. (1969), The simple majority decision rule, *Econometrica 32:490-506*

Kornhauser L. A., Sager L.G. (1986), Unpacking the court, *Yale Law Journal 96:82-117*

Kornhauser L. A., Sager L.G. (2004), Group Choice in paradoxical cases, *Philosophy and Public Affairs 32:249-276*

Laffond G., Lainé J. (2006), Single-switch preferences and the Ostrogorski paradox, *Mathematical Social Sciences 52:49-66*

Laffond G., Lainé J. (2009), Condorcet choice and the Ostrogorski paradox, *Social Choice and Welfare 32:317-333*

Laffond G., Lainé J. (2013), Unanimity and the Anscombe's paradox, *TOP, 21(3):590-611*

List C., Pettit P. (2002), Aggregating sets of judgments: An impossibility result, *Economic Philosophy 18:89-110*

List C., Puppe C. (2009), Judgment aggregation: A survey, in *P. Anand, P. Pattanaik, C. Puppe (eds), Handbook of Rational and Social Choice, Oxford University Press, Oxford*

Maskin E., (1995), Majority rule, social welfare functions, and game forms, *in K. Basu, P. Pattanaik, K. Suzumura (eds), Choice, Welfare, and Development (Essays in honor of Amartya Sen), Oxford University Press, 100-109*

Monjardet B. (2009), Acyclic domains of linear orders: a survey, In: *Brams S., Gehrlein W.V., Roberts F.S. (eds) The Mathematics of Preference, Choice and Order, Essays in honor of Peter C. Fishburn (Studies in Social Choice and Welfare), Springer-Verlag, Berlin, Heidleberg, 139-160*

Nehring K., Puppe C., (2010), Abstract Arrovian aggregation, *Journal of Economic Theory 145:467-494*

Özkal-Sanver I., Sanver M.R., (2006), Ensuring Pareto optimality by referendum voting, *Social Choice and Welfare 27:211-219*

Scarsini M. (1998), A strong paradox of multiple elections, *Social Choice and Welfare 15:237–238*

Sen A.K., Pattanaik P.K. (1969), Necessary and sufficient conditions for rational choice under majority decision, *Journal of Economic Theory 1:178-202*

Wilson R. (1975), On the theory of aggregation, *Journal of Economic Theory 10:89-99*

6 Appendix: Proof of Theorem 3

The sufficiency part follows from Theorem 2 together with Propositions 1 and 2. We prove the necessary part through several intermediate lemmas. Let φ be an anonymous and independent aggregation rule. Independence together with anonymity imply that for any domain $\mathcal{D}$, for any issue $d \in \{1, ..., D\}$, for any $n \geq 3$ and for any admissible $(\mathcal{D}, n)$-profile $\pi_n^{\mathcal{D}} \in \Gamma_n^{\mathcal{D}}$, $[\varphi(\pi_n^{\mathcal{D}})]^d$ only depends on $n_1(\pi_n^{\mathcal{D}} \mid_d)$. In order to lighten notations, we write $\pi = \pi_n^{\mathcal{D}}$ when there is no room for confusion. Consider the set $\Gamma_3^{\mathcal{D}}$ of all admissible profiles which involve exactly 3 individuals. Pick any $\pi \in \Gamma_3^{\mathcal{D}}$ together with an issue d. Then, there exists $a_d \in \{-1, 1\}$ such that for all issues d,

Case 1: $[n_{a_d}(\pi \mid_d) = 2$ and $n_{-a_d}(\pi \mid_d) = 1 \Rightarrow [\varphi(\pi)]^d = a_d]$ (property M^+)

Case 2: $[n_{a_d}(\pi \mid_d) = 2$ and $n_{-a_d}(\pi \mid_d) = 1 \Rightarrow [\varphi(\pi)]^d = -a_d]$ (property M^-)

Case 3: $[n_{-a_d}(\pi \mid_d) \in \{1, 2\} \Rightarrow [\varphi(\pi)]^d = a_d]$ (property $M^=$)

Suppose that for some domain $\mathcal{D}$, φ violates the majority criterion at some issue d. Thus there exist $h > h' > 0$, $d \in \{1, ..., D\}$, $a_d \in \{-1, 1\}$ and $\pi \in \Gamma_{h+h'}^{\mathcal{D}}$ such that $n_{a_d}(\pi \mid_d) = h$, $n_{-a_d}(\pi \mid_d) = h'$ and $[\varphi(\pi)]^d = -a_d$. We denote by h_φ the smallest integer h for which the above situation holds for φ.

Lemma 1 *If $\varphi \neq \varphi_m$ is triple-consistent, then $h_\varphi = 2$.*

Proof Suppose that for some domain $\widetilde{\mathcal{D}}$ there exists an admissible $(\widetilde{\mathcal{D}}, h_\varphi + h')$-profile $\widetilde{\pi}$ such that $n_{a_d}(\widetilde{\pi} \mid_d) = h_\varphi \geq 3$, $n_{-a_d}(\widetilde{\pi} \mid_d) = h'$ and $[\varphi(\widetilde{\pi})]^d = -a_d$ for some issue d. Write $a = a_d$. Define $\mathcal{D} = \{x, y, z\} \in \{-1, 1\}^3$, where $x = (-a, a, -a)$, $y = (a, -a, a)$, and $z = (a, a, a)$. Consider the $(\mathcal{D}, 3)$-profile $\pi = (x, y, z)$. Since $n_a(\pi \mid_d) < h_\varphi$ for all d, then $\varphi(\pi) = z$, and thus $\mathcal{D}$ is 3-stable under φ. Consider the $(\mathcal{D}, h_\varphi + h')$-profile $\pi' = (\overbrace{x, ..., x}^{h'}, \overbrace{y, ..., y}^{h'}, \overbrace{z, ..., z}^{h_\varphi - h'})$. Since for all issues d, $n_a(\pi' \mid_d) = h_\varphi = n_a(\widetilde{\pi} \mid_d)$ and $n_{-a}(\pi' \mid_d) = h' = n_{-a}(\widetilde{\pi} \mid_d)$, then $\pi' \in \Gamma^{\mathcal{D}}_{h_\varphi + h'}$. Moreover, $[\varphi(\pi')]^d = -a$. It follows from independence that $\varphi(\pi') = (-a, -a, -a) \notin \mathcal{D}$, and therefore φ is not triple-consistent. Thus, $h_\varphi \leq 2$. Finally, since $h_\varphi > h' > 0$, then $h_\varphi = 2$ □

An immediate consequence of Lemma 1 is:

Lemma 2 *If φ is triple-consistent and such that property M^+ holds, then for all $\mathcal{D} \in \Delta$, for all $n \geq 3$, and for all non-unanimous profiles $\pi^{\mathcal{D}}_n \in \Gamma^{\mathcal{D}}_n$, $\varphi(\pi^{\mathcal{D}}_n) = \varphi_m(\pi^{\mathcal{D}}_n)$.*

We generalize below Lemma 2 to any admissible profile, unanimous or not. An intermediate step is

Lemma 3 *If φ is triple-consistent and unanimous in restriction to 3-program profiles, then φ is unanimous.*

Proof Define $\mathcal{D} = \{x, y, z, t\} \subset \{-1, 1\}^3$, where $x = (a, a, a)$, $y = (-a, -a, a)$, $z = (a, -a, a)$, and $t = (-a, a, a)$. Since $\mathcal{D}$ contains all programs with a as third coordinate, and since φ is unanimous in restriction to $\Gamma^{\mathcal{D}}_3$, then $\mathcal{D}$ is 3-stable under φ. Thus, triple-consistency ensures that $\mathcal{D}$ is stable under φ. Suppose that $n_a(\widetilde{\pi}^{\widetilde{\mathcal{D}}}_n \mid_d) = n$ and $[\varphi(\widetilde{\pi}^{\widetilde{\mathcal{D}}}_n)]^d = -a$ for some $\widetilde{\mathcal{D}} \in \Delta$, some $n \geq 3$ and some $\widetilde{\pi}^{\widetilde{\mathcal{D}}}_n \in \Gamma^{\mathcal{D}}_n$. It follows from anonymity together with independence that $[\varphi(\pi^{\mathcal{D}}_n)]^d = -a$ for any $\pi^{\mathcal{D}}_n \in \Gamma^{\mathcal{D}}_n$ such that $n_a(\pi^{\mathcal{D}}_n \mid_d) = n$. Thus $\mathcal{D}$ is not stable under φ, in contradiction with triple-consistency □

Lemma 4 *If φ is triple-consistent and satisfies property M^+, then $\varphi = \varphi_m$.*

Proof From Lemma 2 together with Lemma 3, it suffices to prove that φ is unanimous at all profiles $\pi^{\mathcal{D}}_3 \in \Gamma^{\mathcal{D}}_3$. Suppose the contrary. It follows from anonymity and independence that there exists $a \in \{1, -1\}$ such that for all $\pi^{\mathcal{D}}_3 \in \Gamma^{\mathcal{D}}_3$ and all d, $n_a(\pi^{\mathcal{D}}_3 \mid_d) = 3$ and $[\varphi(\pi^{\mathcal{D}}_3)]^d = -a$ (Property A). Define $\mathcal{D} = \{x, y, z, t, -x, -y, -z, -t\} \subset \{-1, 1\}^4$, where $x = (a, a, a, -a)$, $y = (a, a, -a, a)$, $z = (a, -a, a, a)$, and $t = (-a, a, a, a)$. From M^+ together with Property A, we get that $\varphi(x, y, z) = t$, $\varphi(x, y, t) = z$, $\varphi(x, z, t) = y$, and $\varphi(y, z, t) = x$. Furthermore, for any $w \neq w' \neq w'' \in \{x, y, z, t\}$, we have $\varphi(w, w, w') = -w' = \varphi(-w, -w, w')$, $\varphi(w, -w, w') = w'$, $\varphi(w, -w, -w') = -w'$, $\varphi(w, w', w'') = \varphi(-w, -w', w'') \in \mathcal{D}$, and $\varphi(-w, -w', -w'') = -\varphi(w, w', w'') \in \mathcal{D}$. Thus $\mathcal{D}$ is 3-stable under φ. Finally, we get from Lemma 2 that $\varphi(x, y, z, t) = (a, a, a, a) \notin \mathcal{D}$. Thus, φ is not triple-consistent, a contradiction □

We now turn to aggregation rules sharing property $M^=$.

Lemma 5 *If φ is triple-consistent and such that property $M^=$ holds, then φ is almost constant.*

Proof Suppose that for some domain $\widetilde{\mathcal{D}}$, there exist $a \in \{-1, 1\}$ and $h \neq h' \in \mathbb{N}$ such that $h + h' > 3$ and, for some d and some $\pi^{\widetilde{\mathcal{D}}}_{h+h'} \in \Gamma^{\widetilde{\mathcal{D}}}_{h+h'}$, $n_a(\pi^{\widetilde{\mathcal{D}}}_n \mid_d) = h'$, $n_{-a}(\pi^{\widetilde{\mathcal{D}}}_n \mid_d) = h$, and $[\varphi(\pi^{\widetilde{\mathcal{D}}}_n)]^d = -a$. Using independence together with anonymity, we have $\forall d$, $\forall \mathcal{D} \in \Delta$, $[n_a(\pi^{\mathcal{D}}_n \mid_d) = h$, $n_{-a}(\pi^{\mathcal{D}}_n \mid_d) = h'$ $] \Rightarrow [\varphi(\pi^{\mathcal{D}}_n)]^d = -a$. Consider the domain $\mathcal{D}^* \subset \{-1, 1\}^{h+1}$ defined in the table below, where issue-wise positions appear in columns and programs in rows:

Table 1: Domain $\mathcal{D}^*$

$d=$	1	2	...	$d-1$	d	$d+1$	...	h	$h+1$
x_1	a	$-a$	...	$-a$	$-a$	$-a$	...	$-a$	$-a$
x_2	$-a$	a	...	$-a$	$-a$	$-a$	...	$-a$	$-a$
...	...	...	...	...	...	...	...	...	...
x_{d-1}	$-a$	$-a$	...	a	$-a$	$-a$	...	$-a$	$-a$
x_d	$-a$	$-a$	...	$-a$	a	$-a$	...	$-a$	$-a$
x_{d+1}	$-a$	$-a$	...	$-a$	$-a$	a	...	$-a$	$-a$
...	...	...	...	...	...	...	...	...	...
x_h	$-a$	$-a$	...	$-a$	$-a$	$-a$	...	a	$-a$
x_{h+1}	$-a$	$-a$	...	$-a$	$-a$	$-a$	...	$-a$	a
x_{h+2}	a	a	...	a	a	a	...	a	a

Then, we define $\mathcal{D} = \{1,-1\}^{h+1} - \{(-x_{h+2})\}$. Pick up any triple $\{x,y,z\} \subset \mathcal{D}$ such that $\pi = (x,y,z) \in \Gamma_3^{\mathcal{D}}$. From the definition of $\mathcal{D}$, we have that for all d, there exists $w \in \{x,y,z\}$ such that $w^d = -a$. Hence, Property $M^=$ implies that $\varphi(\pi) \neq (-x_{h+2})$. Therefore $\varphi(\pi) \in \mathcal{D}$, and $\mathcal{D}$ is 3-stable under φ. Finally, define the $(\mathcal{D}, h+h')$-profile $\pi^* = (x_1, x_2, ..., x_{h+1}, \overbrace{x_{h+2}, ..., x_{h+2}}^{h'-1})$ that contains only programs in $\mathcal{D}^* \subset \mathcal{D}$. Since $\forall d \in \{1,...,h+1\}$, $n_a(\pi^* \mid_d) = h'$ and $n_{-a}(\pi^* \mid_d) = h$. Since $\pi_{h+h'}^{\widetilde{\mathcal{D}}} \in \Gamma_{h+h'}^{\widetilde{\mathcal{D}}}$ and $n_a(\pi_{h+h'}^{\widetilde{\mathcal{D}}} \mid_d) = h'$ and $n_{-a}(\pi_{h+h'}^{\widetilde{\mathcal{D}}} \mid_d) = h$, then $\pi^* \in \Gamma_{h+h'}^{\mathcal{D}} \Pi_\varphi$. It follows from anonymity together with independence that $\varphi(\pi^*) = -x_{h+2} \notin \mathcal{D}$. Thus, $\mathcal{D}$ is not stable under φ, in contradiction with triple-consistency. Therefore, we have shown that $\forall \mathcal{D} \in \Delta$, $\forall n \geq 3$, $\forall \pi_n^{\mathcal{D}} \in \Gamma_n^{\mathcal{D}}$, $\forall a \in \{-1,1\}$, $0 < n_a(\pi_n^{\mathcal{D}} \mid_d) < n \Rightarrow [\varphi(\pi_n^{\mathcal{D}})]^d = a$, and thus φ is almost constant □

It remains to consider Property M^-.

Lemma 6 *If φ is triple-consistent and such that property M^- holds, then $\varphi(\pi_3^{\mathcal{D}}) = \varphi_m^-(\pi_3^{\mathcal{D}})$ for all $\mathcal{D} \in \Delta$ and all $\pi_3^{\mathcal{D}} \in \Gamma_3^{\mathcal{D}}$.*

Proof It suffices to prove that $\forall a \in \{1,-1\}$, $\forall \pi_3^{\mathcal{D}} \in \Gamma_3^{\mathcal{D}}$, $\forall d$, $n_a(\pi_3^{\mathcal{D}} \mid_d) = 3 \Rightarrow [\varphi(\pi_3^{\mathcal{D}})]^d = -a$. Suppose the contrary, and consider again the domain $\mathcal{D}$ defined in the proof of Lemma 4, where $\forall w \in \mathcal{D}$, $-w \in \mathcal{D}$. Moreover, for any non-unanimous profile $(w, w', w") \in \mathcal{D}^3$, one has from Property M^- that $\varphi(w,w',w'') = -\varphi_m(w,w',w'')$. Since $\mathcal{D}$ is 3-stable under φ_m, then $\mathcal{D}$ is 3-stable under φ. Now consider profile $\widetilde{\pi} = (x,y,z,t)$. Observe that $\forall d$, $n_a(\widetilde{\pi} \mid_d) = 3$ and $n_{-a}(\widetilde{\pi} \mid_d) = 1$. From independence together with anonymity, we get that $\varphi(\widetilde{\pi}) = \{(a,a,a,a), (-a,-a,-a,-a)\}$. Thus $\varphi(\widetilde{\pi}) \notin \mathcal{D}$, therefore φ is not triple-consistent, a contradiction. Finally, independence together with anonymity ensure that $\forall\ \widetilde{\mathcal{D}} \in \Delta$, $\forall \pi_3^{\widetilde{\mathcal{D}}} \in \Gamma_3^{\widetilde{\mathcal{D}}}$, $n_a(\pi_3^{\widetilde{\mathcal{D}}} \mid_d) = 3 \Rightarrow [\varphi(\pi_3^{\widetilde{\mathcal{D}}})]^d = -a$, and the proof is complete □

Lemma 7 *Let φ be triple-consistent and such that property M^- holds. Let $h > h' \in \mathbb{N}$ with $h+h'$ odd. Consider any domain $\mathcal{D}$. Then for any $\pi_{h+h'}^{\mathcal{D}} \in \Gamma_{h+h'}^{\mathcal{D}}$, for any $a \in \{-1,1\}$ and for any issue d,*

(1) if $[n_a(\pi_{h+h'}^{\mathcal{D}} \mid_d) = h$ and $n_{-a}(\pi_{h+h'}^{\mathcal{D}} \mid_d) = h' \Rightarrow [\varphi(\pi_{h+h'}^{\mathcal{D}})]^d = -a]$, then $[n_{-a}(\pi_{h+h'}^{\mathcal{D}} \mid_d) = n \Rightarrow [\varphi(\pi_{h+h'}^{\mathcal{D}})]^d = a]$.

(2) if $[n_a(\pi_{h+h'}^{\mathcal{D}} \mid_d) = h$ and $n_{-a}(\pi_{h+h'}^{\mathcal{D}} \mid_d) = h' \Rightarrow [\varphi(\pi_{h+h'}^{\mathcal{D}})]^d = a]$, then $[n_{-a}(\pi_{h+h'}^{\mathcal{D}} \mid_d) = n \Rightarrow [\varphi(\pi_{h+h'}^{\mathcal{D}})]^d = -a]$.

Proof From Lemma 6, $\varphi = \varphi_m^-$ in restriction to 3-program profiles. Define $\mathcal{D} = \{x,y,z,-x,-y,-z\} \subset \{-1,1\}^3$ by $x = (-a,a,a)$, $y = (-a,a,-a)$ and $z = (-a,-a,a)$. Using Property M^-, it is easily checked that $\mathcal{D}$ is 3-stable under φ. Consider the $(\mathcal{D}, h+h')$-profile $\pi = (\overbrace{x,...,x}^{h-h'}, \overbrace{y,...,y}^{h'}, \overbrace{z,...,z}^{h'})$. Then $n_a(\pi \mid_d) = h$ and $n_{-a}(\pi \mid_d) = h'$ for $d = 2,3$. Suppose that $[\varphi(\pi)]^2 = -a$. From independence and anonymity, we get $[\varphi(\pi)]^3 = -a$. Moreover, triple-consistency requires $\varphi(\pi) = (a,-a,-a)$. Since $n_{-a}(\pi \mid_1) = h+h'$, then we must have $[\varphi(\pi)]^1 = a$, and assertion (1) follows from independence and anonymity. Similarly, if $[\varphi(\pi)]^2 = a$, then $[\varphi(\pi)]^3 = a$, while triple-consistency requires $\varphi(\pi) = (-a,a,a)$. Since $n_{-a}(\pi \mid_1) = h+h'$, then we must have $[\varphi(\pi)]^1 = -a$, and assertion (2) follows from independence and anonymity □

Lemma 8 *Let φ be triple-consistent and such that property M^- holds. For any domain $\mathcal{D}$, for any $n \in 2\mathbb{N}+1$ and any $(\mathcal{D}, n)$-profile $\pi_n^{\mathcal{D}}$, either $\varphi(\pi_n^{\mathcal{D}}) \in \{\varphi_m(\pi_n^{\mathcal{D}}), \varphi_m^-(\pi_n^{\mathcal{D}})\}$.*

Proof Consider any domain $\mathcal{D}$. Suppose first that there exist $a \in \{1, -1\}$ together with $\{h, h', k, k'\} \subset \mathbb{N}$ with $h > h'$, $k > k'$, and $n = h + h' = k + k'$, such that, for two $(\mathcal{D}, n)$-profiles $\pi, \pi' \in \Gamma_n^{\mathcal{D}}$, one has for some issue d:

(1) $n_a(\pi \mid_d) = h$, $n_{-a}(\pi \mid_d) = h'$, and $[\varphi(\pi)]^d = a$

(2) $n_a(\pi' \mid_d) = k$, $n_{-a}(\pi' \mid_d) = k'$, and $[\varphi('\pi)]^d = -a$

It follows from Lemma 7 that for any issue d', $[n_{-a}(\pi \mid_{d'}) = n \Rightarrow \varphi(\pi)]^d = -a]$ and $[n_{-a}(\pi' \mid_{d'}) = n \Rightarrow [\varphi(\pi)]^d = a]$, which clearly contradicts hat φ is independent and anonymous. Thus, we have proved that:

(3) $\forall a \in \{1, -1\}$, $\forall d$, either $[n_a(\pi \mid_d) > n_{-a}(\pi \mid_d)$ and $[\varphi(\pi)]^d = a]$, or $[n_a(\pi \mid_d) > n_{-a}(\pi \mid_d)$ and $[\varphi(\pi)]^d = -a]$.

Next, suppose that there exist $\mathcal{D} \in \Delta$, $n \in \mathbb{N}$ and $a \in \{1, -1\}$ such that for some $\pi \in \Gamma_n^{\mathcal{D}}$ and some d, we have

(4) $[n_a(\pi \mid_d) > n_{-a}(\pi \mid_d) \Rightarrow [\varphi(\pi)]^d = -a]$ and $[n_{-a}(\pi \mid_d) > n_a(\pi \mid_d) \Rightarrow [\varphi(\pi)]^d = -a]$

Consider $\mathcal{D}^* = \{x, y\} \subset \{-1, 1\}^2$, with $x = (a, -a)$ and $y = (-a, a)$. It follows from Lemma 6 that $\varphi(x, x, y) = y$ and $\varphi(x, y, y) = x$, and thus $\mathcal{D}^*$ is 3-stable under φ. Then define the $(\mathcal{D}^*, t + t')$-profile $\pi = (\overbrace{x, ..., x}^{t}, \overbrace{y, ..., y}^{t'})$, where $t > t'$. One gets from independence together with (4) that $\varphi(\pi) = (-a, -a) \notin \mathcal{D}^*$, therefore φ is not triple-consistent, a contradiction. Thus, we have either $[n_a(\pi \mid_d) > n_{-a}(\pi \mid_d) \Rightarrow [\varphi(\pi)]^d = -a]$ or $[n_a(\pi \mid_d) > n_{-a}(\pi \mid_d) \Rightarrow [\varphi(\pi)]^d = -a]$. Using anonymity together with independence, this proves that for all $\mathcal{D} \in \Delta$, for all d, for all $n \in 2\mathbb{N}+1$ and all $a \in \{1, -1\}$, and for all $\pi_n^{\mathcal{D}} \in \Pi_n^{\mathcal{D}}$,

either

(5) $[n_a(\pi_n^{\mathcal{D}} \mid_d) > n_{-a}(\pi_n^{\mathcal{D}} \mid_d) \Rightarrow [\varphi(\pi_n^{\mathcal{D}})]^d = a]$ and $[n_{-a}(\pi_n^{\mathcal{D}} \mid_d) > n_a(\pi_n^{\mathcal{D}} \mid_d) \Rightarrow [\varphi(\pi)]^d = -a]$

or

(6) $[n_a(\pi_n^{\mathcal{D}} \mid_d) > n_{-a}(\pi_n^{\mathcal{D}} \mid_d) \Rightarrow [\varphi(\pi_n^{\mathcal{D}})]^d = -a]$ and $[n_{-a}(\pi_n^{\mathcal{D}} \mid_d) > n_a(\pi_n^{\mathcal{D}} \mid_d) \Rightarrow [\varphi(\pi)]^d = a]$

Since (5) defines φ_m while (6) defines φ_m^-, the proof is complete □

The proof of Theorem 3 is now complete. Indeed, consider any anonymous, independent and triple-consistent aggregation rule φ. If M^+ holds, then Lemma 4 implies that $\varphi = \varphi_m$. If $M^=$ holds, then, from Lemma 5, φ is almost constant. If M^- holds, Lemma 8 imply that $\varphi \in \Phi_m^{+-}$.