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Abstract In this paper, we theoretically analyze the effect of the wealth levels and underlying bankruptcy rules on investment decisions in a bankruptcy game setting. There is considerable literature dealing with the question of which principle bankrupt values should be divided according to and studying the axiomatic properties of different bankruptcy rules. However, very few studies focus on the relation between the underlying bankruptcy rule to be used and the investment decisions, and this study aims to contribute to this literature. More specifically, we develop a bankruptcy model that enables us to study the wealth effect, using DARA (Decreasing Absolute Risk Aversion) as investors' utility function. This utility function specification enables us to include the uninvested portion of their wealth in the utility equation and reach intuitive equilibrium behavior. By using Nash Equilibrium as the solution concept, investment levels are reviewed under different rules and parameters. We focus on the three commonly studied bankruptcy rules in the literature, namely, Proportionality (PRO), Equal Awards (EA), and Equal Losses (EL). These rules are examined separately and as combinations to see which rule(s) leads to higher total investment levels. It is shown that an agent's equilibrium investment is affected by her own wealth and the wealth of the other agents. There is a two agents case for computational and illustrative purposes in the last part of the paper to complement the theoretical part.

Key words : Bankruptcy Rules · Wealth Effect · DARA· Total Investment

1 Introduction

Bankruptcy problems have made their debut in the literature in the 1980s. The pioneering study is the work of O'Neill (1982), which examines a story from Talmud. In the story, a man dies bequeathing a certain amount of estate that needs to be arbitrated between his children. The problem is that the total claims exceed the value of the estate. The large class of such problems, where the asset to be allocated does not fulfill the sum of claims, constitute the class of bankruptcy problems. A specific example can be a firm where each creditor holds a claim and the total value of the claims exceeding the firm's liquidation value.

There are studies in the axiomatic literature providing and analyzing bankruptcy principles such as Aumann (1985), Dagan (1996), Herrero and Villar (2002). An extensive review of the axiomatic literature may be found in Moulin (2002), Thomson (2003), and Thomson (2015). The prominent rules studied are Proportionality (PRO), Equal Awards (EA), Equal Losses (EL), and the constrained versions of the last two of them (CEA, CEL). Aumann (1985), Curiel (1987), and Dagan and Volij (1993) employ cooperative games and find game theoretical solutions to them. On the other hand, Chun (1989), Dagan et al. (1997) use a non-cooperative game-theoretical approach to study the Nash equilibria of the bankruptcy games induced by these prominent rules.

Another approach in this literature uses the strategical approach, in which the value of the asset is endogeneous. In most cases, the value of the asset and the possibly bankrupt value is formed after an investment process where agents make strategical investment decisions. When the rules such as Equal Losses, Equal Awards, or their constrained versions are chosen as a division rule, at the end of this project, one's outcome may be affected by

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others' investment decisions. This relationship makes the effect of bankruptcy principles on investment decisions interesting and valuable. Also, one's wealth might affect the amount of investment she can make. The wealth of others in a common project gain importance once again thanks to the nature of the rules mentioned above. The question of who are we investing together in the same company or project gains importance since after all wealth of other people might affect my outcome with the specified indirect route.

The two studies investigating the implications of bankruptcy principles on total investment levels in a strategical perspective are Karagözoğlu (2014) and Kıbrıs and Kıbrıs (2013). Karagözoğlu (2014) designs a non-cooperative game with two types of agents (high and low income) and analyses the consequences under proportional (PRO), constrained equal awards (CEA), and constrained equal losses (CEL). A fundamental assumption in the model is that the agents are risk-neutral, and this assumption induces corner solutions. That is, each agent chooses either zero investment or invests all her income. As a result, Karagözoğlu (2014) finds that PRO is the total investment maximizing rule.

Kıbrıs and Kıbrıs (2013), is the study more closely related to ours. They employ a non-cooperative game in which agents are assumed to be risk-averse. In the model, agents have Constant Absolute Risk Aversion (CARA) risk preferences. Thus, when everything else remains constant, a change in wealth does not lead to a change in equilibrium investment levels of agents. Parallel with the choice of CARA, they create a model where agents can borrow unlimited money from a bank with an interest rate normalized to 1. Agents borrow the amount of money corresponding to their equilibrium investment level, and after the end of the project, they pay the exact borrowed amount back. The model also assumes each agent has the same credibility. In Kıbrıs and Kıbrıs (2013), Nash Equilibria analysis is made for proportional (PRO), equal awards (EA), and equal losses (EL). EL is singled out as a rule yielding the maximum total equilibrium investment among them. Additionally, they also performs a welfare analysis.

Our model differs from their work in this matter; thanks to the utility function, wealth becomes relevant, and agents react to the wealth changes. We consider agents endowed with Decreasing Absolute Risk Aversion (DARA) preferences. DARA preferences enable us to study the effect of changes in investors' wealth on all agents' investment levels. Thus, agents are endowed with some level of wealth and they are expected to invest a non-negative amount that cannot exceed their wealth.

We consider a simultaneous moves non-cooperative game of investment, and the underlying solution concept is Nash equilibrium. Thanks to preference specification, the agents' wealth level becomes relevant and affects their investment decisions. This impact depends on the underlying bankruptcy rule to be implemented if the investment ends up a failure and the remaining value of the assets is to be divided among the agents. PRO, EL, EA, and mixture rules of the latter two with PRO weighted by $\alpha \in [0,1]$, are analyzed in terms of both equilibrium investment and total equilibrium investment. If the agents' wealth levels increase, it turns out that the equilibrium investment also increases.

DARA type preferences are backed up by evidence in many studies in the experimental literature. Hamal and Anderson (1982) find experimental evidence for DARA among farmers in Nepal. Levy (1994) employs a dynamic portfolio choice experiment. The proportion of assets modifiable in every round. After regressing the amount of risky investment on wealth, he finds that subjects exhibit DARA preferences. Brocas et al. (2019) assume individuals' utility functions belong to a very comprehensive broad family of functions in a more recent study. For different risk aversion parameters, this function becomes CARA, DARA, or IARA, and CRRA, DRRA, or IRRA. They set up an investment game with one safe and one risky asset and ask people to allocate their wealth dynamically between the assets. The main result of the paper is that most of the subjects show DARA and IRRA type preferences.

Our main finding is that an increase in wealth leads to an increase in investment regardless of the underlying bankruptcy rule to be used. There is also another effect that results from the changes in wealth of the other people. This second effect varies with the bankruptcy rules and will be examined in detail throughout the following sections. Finally, similar to Kıbrıs and Kıbrıs (2013), the ranking of the bankruptcy rules regarding total equilibrium investment is EL>PRO>EA in our model.

Rest of the paper is organized as follows. In Section 2, we explain the model. In Section 3, the Nash equilibrium analysis for each of the bankruptcy rules is conducted. We analyze the relation between the total equilibrium investment and the choice of bankruptcy rules in Section 4. In Section 5, we provide a computational illustration of the two-agent case. Finally, in Section 6, we conclude with closing remarks summarizing our results.

2 The Model

Let $N = \{1, ..., n\}$ denote the set of agents interpreted as potential investors, where $n \geq 2$. Each agent $i \in N$ is endowed with the following Decreasing Absolute Risk-Aversion (DARA) utility function, $u_i(x) = \frac{1-\gamma_i}{\gamma_i}(\frac{x}{1-\gamma_i})^{\gamma_i}, \forall x \in R_+$ where $\gamma_i < 1$ for all $i \in N$. This function belongs to the class of Hyperbolic Absolute Risk-Aversion (HARA) type utilities $u(x) = \frac{1-\gamma}{\gamma}(\frac{x}{1-\gamma} + \eta)^{\gamma}$ first used in Merton (1971) for a dynamic portfolio allocation problem. For values of $\gamma < 1$, the function exhibits DARA, that is, as the wealth level of the agent increases, she will be willing to put more money at risk in absolute terms. HARA class of functions also exhibits increasing, constant, and decreasing relative risk aversion for $\eta > 0$, $\eta = 0$ and $\eta < 0$, respectively. We assume constant relative risk aversion, $\eta_i = 0$ for all $i \in N$, to simplify the functional form of our results in this paper. Nonetheless, it should be noted that this assumption does not drive our results. Finally, without loss of generality, we assume that $\gamma_1 \geq \ldots \geq \gamma_n$. Thus, we assume that the level of risk aversion of agents increases with that index. Furthermore, it might be of interest to note that the natural logarithm (ln) utility specification, i.e. $u_i(x) = \ln(x)$, corresponds to the case of $\lim_{\gamma \to 0}$.

Each agent i is endowed with initial wealth $w_i \in R_+$ and simultaneously decides how much wealth to invest on a risky project. We denote the vector of wealths of all the agents by $w = (w_1, \ldots, w_n)$. Let $s_i \in [0, w_i]$ denote the investment of agent i. The vector of investment of all agents is denoted by $s = (s_1, \ldots, s_n)$, and following the investments, the total value of the project becomes S which is equal to the total value of the investments of the agents, $\sum_N s_i$. We let w_{-i} (resp. s_{-i}) denote the wealth (resp. investment) vector of all agents other than i, and with a slight abuse of notation we use N_{-i} (resp. S_{-i}) to denote $N \setminus \{i\}$ (resp. $\sum_{N \setminus \{i\}} s_i$).

With success probability $p \in (0, 1)$, the project brings a return $r \in (0, \bar{r}]$ where $\bar{r} \geq 1$, and the project's value becomes $(1 + r)S$. If the project is successful, the resulting value, $(1 + r)S$, is shared between the agents proportionally to their investments. Thus, an agent i with initial wealth w_i, would obtain $(w_i - s_i) + (1+r)S\left(\frac{s_i}{S}\right) = w_i + rs_i$ if the project succeeds. With the remaining probability $(1 - p)$, the project goes bankrupt, and only the $\beta \in (0, 1)$ fraction of the total value survives. That is, the remaining total value becomes βS. If the project goes under, the firm's value is allocated among the agents according to a prespecified bankruptcy rule.

In this paper, the three most commonly studied bankruptcy rules and their convex combinations are examined. Proportionality (PRO) implies that every investor receives money according to the ratio of her share in the firm. Under PRO, an agent's return is equal to $PRO_i(s) = \beta S\left(\frac{s_i}{S}\right) = \beta s_i$. The second rule is Equal Awards (EA), which implies that, following bankruptcy, every investor shares what is left from the firm equally. Under EA, an agent's return is $EA_i(s) = \frac{\beta}{n}S$. In the division phase of a bankrupt firm, EA is in favor of the smaller investor(s). The last rule we consider is Equal Losses (EL), which implies that the loss that occurred, $(1 - \beta)S$, is shared equally among participants. Under EL, an agent's return is $EL_i(s) = s_i - \frac{(1-\beta)}{n}S$ from a bankrupt project. Since investors divide the loss occurred equally, the division ends in favor of bigger investor(s).[1] Given any $\alpha \in [0, 1]$, the mixture applications of EA-PRO ($AP[\alpha]$) and EL-PRO ($LP[\alpha]$) are constructed by assigning weight α to PRO and the remaining weight $(1 - \alpha)$ to EA (resp. EL). Thus, the return in case of bankruptcy for $AP[\alpha]$ is $AP[\alpha]_i(s) = \alpha PRO_i(s) + (1-\alpha)EA_i(s) = \alpha\beta s_i + (1-\alpha)\frac{\beta}{n}S$. Similarly, the return in case of bankruptcy for $LP[\alpha]$ is $LP[\alpha]_i(s) = \alpha PRO_i(s) + (1 - \alpha)EL_i(s) = \alpha\beta s_i + (1 - \alpha)\left[s_i - \frac{(1-\beta)}{n}S\right]$.[2] Thus, to summarize, the expected utilites of an agent i with wealth level w_i at an investment profile s when the underlying rule to be applied in case of bankruptcy is respectively PRO, $AP[\alpha]$, and $LP[\alpha]$ are given by

$$
\begin{aligned}
U_i^{PRO}(s) &= pu_i(w_i + rs_i) + (1-p)u_i\left[w_i - (1-\beta)s_i\right], \\
U_i^{AP[\alpha]}(s) &= pu_i(w_i + rs_i) + (1-p)u_i\left[(w_i - s_i) + \alpha\beta s_i + (1-\alpha)\frac{\beta}{n}S\right], \text{ and} \\
U_i^{AP[\alpha]}(s) &= pu_i(w_i + rs_i) + (1-p)u_i\left((w_i - s_i) + \alpha\beta s_i + (1-\alpha)\left[s_i - \frac{(1-\beta)}{n}S\right]\right).
\end{aligned}
$$

Remark 1 We restrict the range of parameter values to ensure that for any $\alpha \in [0, 1]$, at any equilibrium investment levels s^*, $AP[\alpha]_i(s^*) \leq s_i^*$ and $LP[\alpha]_i(s^*) \geq 0$. That is, the two rules conincide with their constrained versions. It should be noted that this also guarantees non-negative values of total money under $LP[\alpha]$ in case of bankruptcy. Thus, the expected utilities are well-defined.

3 Analysis of Bankruptcy Principles

In this section, we analyze the Nash equilibria and dominant strategy equilibria of the investment games corresponding to cases in which different prespecified bankruptcy rules are implemented

[1] The well-known constrained version of EA and EL, respectively Constranied Equal Awards (CEA) and Constrained Equal Losses (CEL) are defined as follows. $CEA_i(s) = \min\{EA_i(s), s_i\}$, thus no agent may recive a return greater than her investment. Similarly, $CEL_i(s) = \max\{0, EL_i(s)\}$, thus no agent may recive a negative return.

[2] It is easy to see that for $\alpha = 1$, both $AP[\alpha]$ and $LP[\alpha]$ reduces to PRO. Similarly for $\alpha = 0$, both $AP[\alpha]$ reduces to EA and $LP[\alpha]$ reduces to EL.

3.1 Proportionality (PRO)

The following proposition shows that under Proportional rule (PRO) the investment game has a unique dominant strategy equilibrium:

Proposition 1 *If $pr \leq (1-p)(1-\beta)$, the investment game under the rule PRO has a unique dominant strategy equilibrium $(0,\dots,0)$. Otherwise, the game has a unique dominant strategy equilibrium s^* in which each agent i chooses a positive investment level s_i^* is given by*

$$s_i^* = \frac{\left(1 - \left[\frac{pr}{(1-p)(1-\beta)}\right]^{\frac{1}{\gamma_i - 1}}\right) w_i}{\left[\frac{pr}{(1-p)(1-\beta)}\right]^{\frac{1}{\gamma_i - 1}} r + (1-\beta)}.$$

Proof In the appendix.

It is worth reemphasizing that, for $s_i^* > 0$ to be the unique dominant strategy equilibrium, $pr > (1-p)(1-\beta)$ should hold, which can be interpreted as follows. The left-hand side of the inequality is the expected return on one unit of investment, and the right-hand side is the expected loss of the agent on one unit of investment. It may also be noted that w_i is also positive by definition.

Another comment that directly follows from the above proposition is that for PRO, the optimal investment level s_i^* increases (decreases) as w_i increases (decreases). It is because individuals have DARA utility preferences, as their wealth increases, they become less risk-averse than before and are willing to put more money at risk. It is also worth noting that one could reinterpret this observation to consider an interpersonal comparison of two agents with the same γ values (which ensures that in case of having equal wealth both agents will be equally risk averse) and different levels of wealth. As a final remark, we note that an agent's investment decision is not affected by their opponents' wealth levels or risk attitudes under PRO. That is, given any agent $i \in N$ any change in the wealth levels or risk parameters of other agents does not lead to a change in the optimal investment level s_i^*.

3.2 EA-PRO Mixture Rule - $AP[\alpha]$

The following proposition determines the form of the unique Nash equilibrium under $AP[\alpha]$. We also consider the restriction on the model's parameter values so that at the Nash equilibrium, an agent's compensation in case of bankruptcy is no more than his investment, and no agent invests more than her wealth. Thus, we also consider as an additional constraint that the parameter values are such that $AP[\alpha]_i(s^*) \leq s_i^*$ and $w_i \geq s_i^* \geq 0$ for each $i \in N$. It should be noted that we have numerically shown that range of such parameter values is large enough, that is, even under this additional constraints, the model is reasonably rich.

Proposition 2 *If $pr \leq (1-p)\frac{[n-\beta-(n-1)\alpha\beta]}{n}$, the investment game under the rule PRO has a unique Nash equilibrium $(0,\dots,0)$. Otherwise, the game has a unique Nash equilibrium s^* in which each agent i chooses a positive investment level s_i^* given by*

$$s_i^* = \frac{(1-A_i)\, w_i \left(\prod_N (A_i r + \delta) - C\sum_N \left[\prod_{N_{-i}} (A_j r + \delta)\right]\right) + C\sum_N \left[(1-A_i) w_i \prod_{N_{-i}} (A_j r + \delta)\right]}{[A_i r + \delta]\left(\prod_N (A_i r + \delta) - C\sum_N \left[\prod_{N_{-i}} (A_j r + \delta)\right]\right)},$$

where $A_i = \left[\frac{npr}{(1-p)[n-\beta-(n-1)\alpha\beta]}\right]^{\frac{1}{\gamma_i - 1}}$, $C = (1-\alpha)\frac{\beta}{n}$, and $\delta = 1 - \alpha\beta$, under the additional constraints that $AP[\alpha]_i(s^*) \leq s_i^*$ and $w_i \geq s_i^* \geq 0$.

Proof In the appendix.

Thus, to have $s_i^* > 0$ as the unique equilibrium investment level, $pr > (1-p)\frac{[n-\beta-(n-1)\alpha\beta]}{n}$ should hold. We could explain the left-hand side as the expected return on one unit of investment, and the right-hand side is the expected loss of the agent on one unit of investment.

Remark 2 It is worth noting that $pr > (1-p)\frac{[n-\beta-(n-1)\alpha\beta]}{n}$ and $\prod_N [A_i r + (1-\alpha\beta)] > C\sum_N \left(\prod_{N_{-i}} [A_j r + (1-\alpha\beta)]\right)$ are sufficient conditions for $s_i^* > 0$ in the general case with any number of agents. For the special case of $n = 2$, i.e. only two agents, $s_i^* > 0$ follows from $pr > (1-p)\frac{[n-\beta-(n-1)\alpha\beta]}{n}$. Put differently, for the case of $n = 2$, $\prod_N [A_i r + (1-\alpha\beta)] > C\sum_N \left(\prod_{N_{-i}} [A_j r + (1-\alpha\beta)]\right)$ condition is automatically satisfied as long as $pr > (1-p)\frac{[n-\beta-(n-1)\alpha\beta]}{n}$. Computationally, we have shown that the same result holds for $n = 3$. Nonetheless, due to complicated nature of the solution, it seems as a difficult problem to check whether one of the conditions implies the other for any number of agents.

In the following subsection, we restrict our attention to the analysis of equilibria under $AP[\alpha]$ rule, under the simplifying assumption of only two agents with a shared risk aversion parameter.

3.2.1 Two-agent Case This is a miniature version of the model with two investors, $i = 1, 2$, with equal risk aversion parameters ($\gamma_1 = \gamma_2 = \gamma$). The equilibrium investment level is obtained with the same procedure of n investor case. The reason this exercise seems relevant in our view is that there are many variables in n agent case affecting the optimal investment level, such as γ_i, w_i for $i \in N$. With this example and these assumptions, we will be able to analyze both total investment comparison for rules and see the effect of wealth more clearly.

Under the simplifying assumptions, we get:
$s_1^* = \frac{B_1[F-(1-\alpha)\beta]+(1-\alpha)\frac{\beta}{2}(B_1+B_2)}{F^2-(1-\alpha)\beta F}$, where
$A = \left[\frac{2pr}{(1-p)(2-\beta-\alpha\beta)}\right]^{\frac{1}{\gamma-1}}$, $B_i = (1-A)(w_i)$ for $i = 1, 2$, $C = (1-\alpha)\frac{\beta}{2}$, and $F = Ar + (1-\alpha\beta)$.
Thus,

$$s_1^* = \frac{(1-A)w_1\left[Ar + (1-\beta)\right] + (1-\alpha)\frac{\beta}{2}(1-A)\left(w_1 + w_2\right)}{\left[Ar + (1-\alpha\beta)\right]\left[Ar + (1-\beta)\right]}.$$

All parts but $1 - A$ in the optimal investment level are positive. So the unique condition for equilibrium investment level to be strictly positive is $1 - A > 0$. And as in the n agent case, this condition reduces to $pr > (1-p)\frac{[n-\beta-(n-1)\alpha\beta]}{n}$.

The total investment $S = \frac{B_1F+B_2F}{F^2-(1-\alpha)\beta F} = \frac{B_1+B_2}{F-(1-\alpha)\beta} = \frac{(1-A)[w_1+w_2+2(1-\gamma)\eta]}{Ar+(1-\beta)}$.

As in the similar exercise for *PRO*, the equilibrium investment is increasing on w_1 under $AP[\alpha]$. Moreover, w_2 has an effect too. The reason for the presence of those variables in the formula is the DARA utility function assigned to each agent. For the broad range of $\gamma < 1$, investors with any risk aversion degree exhibit the same behavior in relation to both their own wealth and the wealth of others. The wealth of the other investors have an effect on s_1 through the idea that for different wealth level of opponents', the amount of their investments vary. Further analysis of what happens when w_1 or w_2 increase (decrease) will take place in Section 4.

In order to make sure that no agent can earn more than her investment when she chooses to invest the optimal investment level, $s_i^* \geq \alpha\beta s_i^* + (1-\alpha)\frac{\beta}{2}S$ condition is necessary. The last necessary condition to be checked is $w_i \geq s_i^*$. For some small interval of values of parameters given, the optimal investment level might be greater than the endowment level of the agent. Since the whole process is an unconstrained optimization, this constraint has to be regarded exclusively.

3.3 EL-PRO Mixture Rule - LP[α]

The following proposition determines the form of the unique Nash equilibrium under $LP[\alpha]$. We also consider the restriction on the model's parameter values so that at the Nash equilibrium, an agent's compensation in case of bankruptcy is nonnegative, and no agent invests more than her wealth. Thus, we also consider as an additional constraint that the parameter values are such that which $0 \leq LP[\alpha]_i(s^*)$ and $w_i \geq s_i^* \geq 0$ for each $i \in N$. It should be noted that we have numerically shown that range of such parameter values is large enough, that is, even under this additional constraints, the model is reasonably rich.

Proposition 3 *If* $pr \leq (1-p)\frac{(1-\beta)[1+(n-1)\alpha]}{n}$*, the investment game under the rule PRO has a unique Nash equilibrium* $(0, \ldots, 0)$*. Otherwise, the game has a unique Nash equilibrium* s^* *in which each agent* i *chooses a positive investment level* s_i^* *is given by*

$$s_i^* = \frac{(1-A_i)w_i\left(\prod_N (A_i r + \delta) + C\sum_N \left[\prod_{N_{-i}}(A_j r + \delta)\right]\right) - C\sum_N\left[(1-A_i)w_i\left(\prod_{N_{-i}}(A_j r + \delta)\right)\right]}{(A_i r + \delta)\left(\prod_N (A_i r + \delta) + C\sum_N\left[\prod_{N_{-i}}(A_i r + \delta)\right]\right)},$$

where $A_i = \left[\frac{npr}{(1-p)(1-\beta)[1+(n-1)\alpha)]}\right]^{\frac{1}{\gamma_i - 1}}$, $C = \frac{(1-\alpha)(1-\beta)}{n}$, $\delta = (1-\beta)\alpha$*, under the additional constraints that* $0 \leq LP[\alpha]_i(s^*)$ *and* $w_i \geq s_i^* \geq 0$.

Proof In the appendix.

So to have $s_i^* > 0$ as equilibrium investment level, $pr > (1-p)\frac{(1-\beta)[1+(n-1)\alpha]}{n}$ should hold. We could explain the left-hand side as the expected return on one unit of investment, and the right-hand side is the expected loss of the agent on one unit of investment. In order to have every investor not suffering losses more than their investment, s_i^* should be greater or equal to the loss anyone faces in the case of bankruptcy. That is, $s_i^* \geq \alpha(1-\beta)s_i^* - (1-\alpha)\frac{(1-\beta)}{n}S$. The last condition we should keep in mind is that the optimal investment level should not exceed wealth, that is, $w_i \geq s_i^*$.

3.3.1 Two-agent Case Now we consider a miniature version of the model with two agents, $i = 1, 2$, experiencing equal risk aversion parameters, i.e., $\gamma_1 = \gamma_2 = \gamma$. The equilibrium investment follows from our previous result on n investor case, proposition 3. The reason we are analyzing this simplified version is the intractability of the general model. Under these assumptions, we will be able to analyze both total investment comparisons for rules and see the effect of wealth.

This time, using a similar notation as above:
$A = \left[\frac{2pr}{(1-p)(1-\beta)(1+\alpha)}\right]^{\frac{1}{\gamma-1}}$, $B_i = (1-A)(w_i)$ for $i = 1, 2$, $C = \frac{(1-\alpha)(1-\beta)}{2}$, $F = Ar + \alpha(1-\beta)$,
$s_1^* = \frac{B_1[F+(1-\alpha)(1-\beta)] - \frac{(1-\alpha)(1-\beta)}{2}(B_1+B_2)}{F^2+(1-\alpha)(1-\beta)F}$, which simplifies to

$$s_1^* = \frac{(1-A)(w_1)\left[Ar + (1-\beta)\right] - \frac{(1-\alpha)(1-\beta)}{2}(1-A)(w_1+w_2)}{\left[Ar + \alpha(1-\beta)\right]\left[Ar + (1-\beta)\right]}.$$

With a similar exercise, we can only talk about a positive investment where $1 - A$ is already positive. After that is satisfied, the condition for s_1^* to be a positive equilibrium is $B_1\left[F + (1-\alpha)(1-\beta)\right] > \frac{(1-\alpha)(1-\beta)}{2}(B_1 + B_2)$.

So to have $s_i^* > 0$ for $i = 1, 2$ as equilibrium investment level, $pr > (1-p)\frac{(1-\beta)(1+\alpha)}{2}$ should hold. As before, one can interpret the left-hand side as the return on unit investment when the firm succeeds, the right-hand side as the loss agents face in the case of bankruptcy.

The total investment $S = \frac{B_1F + B_2F}{F^2-(1-\alpha)\beta F} = \frac{B_1+B_2}{F-(1-\alpha)\beta} = \frac{(1-A)(w_1+w_2)}{Ar+(1-\beta)}$.

For agent i, different wealth levels or changes in wealth induce different levels of equilibrium investment, s_i^*. The effect of each agent's own wealth is increasing on the investment. The wealth of the other investors has an effect on i's investment too, but this time it has a negative effect. The logic behind this is $(1-\alpha)$ share of the loss incurred will be suffered equally by investors. According to the model, if an opponent has greater wealth than before, she would invest more, and now the smaller investors will be facing this danger of sharing equally a greater total loss. So, they decrease their investment in order to prevent losing more and more in case of bankruptcy.

4 Comparison of Principles - Total Equilibrium Investment

We examine mixed rules, $AP[\alpha]$ and $LP[\alpha]$, to determine which principle induces the highest total investment and which one induces the lowest. We restrict our attention to the 2 agent case for the sake of simplicity on expressions where both agents have the same absolute risk aversion parameter, i.e., $\gamma_1 = \gamma_2 = \gamma$. It should be noted that this approach allows us to compare not only pure PRO, EA, and EL among themselves but also talk about the effect of changing the weight, α. Let us start with the comparison of EA and PRO through $AP[\alpha]$.

Proposition 4 (EA vs. PRO - $AP[\alpha]$**)** *PRO leads to weakly higher equilibrium total investment than EA , i.e. PRO ≥ EA. Furthermore, at any parameter values that leads to strictly positive investments at equilibrium, the inequality is strict, i.e. PRO > EA.*

Proof There are three cases to consider:

Case 1) If $pr > \frac{(1-p)(2-\beta-\alpha\beta)}{2}$, $(1 - A > 0)$, $\forall\alpha \in [0, 1]$, for each $AP[\alpha]$ principle, all investors have positive equilibrium investment levels. In this case:
$S = \frac{(1-A)(w_1+w_2)}{Ar+(1-\beta)} = \frac{\left(1-\left[\frac{2pr}{(1-p)(n-\beta-\alpha\beta)}\right]^{\frac{1}{\gamma-1}}\right)(w_1+w_2)}{\left[\frac{2pr}{(1-p)(n-\beta-\alpha\beta)}\right]^{\frac{1}{\gamma-1}}r+(1-\beta)}$,

where S is the total equilibrium investment. If we take partial derivative of S with respect to α; $\frac{\partial S}{\partial \alpha} > 0$ is the result. Therefore, as α increases, share of PRO increases and equilibrium total investment under $AP[\alpha]$ also increases. When the constraints for positive investment are satisfied, PRO yields greater total investment than EA.

Case 2) If $pr = \frac{(1-p)(2-\beta-\alpha\beta)}{2}$ for some $\alpha^* \in [0, 1]$, since the term is increasing in α, the first case applies $\forall\alpha \in (\alpha^*, 1]$. On the other hand, $\forall\alpha \in [0, \alpha^*)$ one has $pr < \frac{(1-p)(2-\beta-\alpha\beta)}{2}$ as in case 3, and all of these levels induce zero investment.

Case 3) If $pr < \frac{(1-p)(2-\beta-\alpha\beta)}{2}$ $\forall\alpha \in [0, 1]$, all $AP[\alpha]$ rules induce zero investment.

We can conclude with the result $PRO > EA$ at any parameter values which leads to positive investments, and $PRO \geq EA$ in general.□

Similarly, the comparison of EL and PRO is carried by taking the derivative of total investment under $LP[\alpha]$ w.r.t. α, and we get the following proposition.

Proposition 5 (EL vs. PRO - $LP[\alpha]$**)** *EL leads to weakly higher equilibrium total investment than PRO , i.e. EL ≥ PRO. Furthermore, at any parameter values that leads to strictly positive investments at equilibrium, the inequality is strict, i.e. EL > PRO.*

Proof As in the proof of preceeding proposition, there are three cases to consider:

Case 1) If $pr > \frac{(1-p)(1-\beta)(1+\alpha)}{2}$ $\forall \alpha \in [0,1]$, for each $LP[\alpha]$ principle, all investors have positive equilibrium investment levels. In this case:

$$S = \frac{(1-A)(w_1+w_2)}{Ar+(1-\beta)} = \frac{\left(1-\left[\frac{npr}{(1-p)(1-\beta)(1+(n-1)\alpha)}\right]^{\frac{1}{\gamma-1}}\right)(w_1+w_2)}{\left[\frac{npr}{(1-p)(1-\beta)(1+\alpha)}\right]^{\frac{1}{\gamma-1}}r+(1-\beta)},$$

where S is the total equilibrium investment. If we take partial derivative of S with respect to α; $\frac{\partial S}{\partial \alpha} < 0$ is the result. Therefore, as α increases, share of PRO increases and the equilibrium total investment under $LP[\alpha]$ decreases. When the constraints for positive investment are satisfied, EL yields greater total investment than PRO.

Case 2) If $pr = \frac{(1-p)(1-\beta)(1+\alpha)}{2}$ for some $\alpha^* \in [0,1]$, since the term is increasing in α, the first case applies $\forall \alpha \in (\alpha^*, 1]$. On the other hand, $\forall \alpha \in [0, \alpha^*)$ the term $pr < \frac{(1-p)(1-\beta)(1+\alpha)}{2}$ and all of these levels induce zero investment.

Case 3) If $pr < \frac{(1-p)(1-\beta)(1+\alpha)}{2}$ $\forall \alpha \in [0,1]$, all $AP[\alpha]$ rules induce zero investment.

We can conclude with the result $EL > PRO$ at any parameter values which leads to positive investments, and $EL \geq PRO$ in general.□

Before proceeding to next section, where we provide a computational illustration of the two-agent case, let us summarize our results concerning comparison of the rules in terms of total equilibrium investment levels. As a straightforward corollary of the preceeding two propositions, we get the following corollary.

Corollary 1 *The ranking of principles in terms of equilibrium total investment is* $EL \geq PRO \geq EA$. *Furthermore, at any parameter values that leads to strictly positive investments at equilibrium, the inequalities are strict, i.e.* $EL > PRO > EA$.

Proof The results follows directly from conjuction of the two preceeding propositions.

5 PRO vs. EA vs. EL
Illustrations of total investment comparisons obtained via computations

Let $n = 2$, $\beta = 0.6$, $p = 0.5$, $r = 2$, $\gamma_1 = \gamma_2 = -1$, and $\alpha = 0$ means the principles will be pure EA and EL. Initially, both agents have DARA utility function with equal γ values, equal wealth, and hence they are equally risk-averse. Therefore, they are expected to yield the same investment levels at equilibrium. We will investigate the changes occurring in the equilibrium investment level of agent 1 and 2, s_1^*, s_2^*, under three principles. The story here is that wealth changes while everything else remains constant. However, wealth change is not necessary to be actualized. A comparison of two different wealth levels would also be enough, and our finding still applies.

Findings from Section 2 are as follows:

1. Wealth is a determinant of the amount of investment, and if w_1 rises, that leads to a rise in s_1^*, vice versa. Underlying reasoning was explained before, by the assumption made with the agent's utility function, individual risk aversion changes with wealth or takes different values for different wealth levels.

2. When n agents invest in a project together under EA or EL, the wealth of other investors affects s_1^*. A change in an opponent's wealth triggers an increase or decrease in one's investment, thanks to DARA. Directions of this reaction will be analyzed with computations and figures.

3. In the equilibrium, where $n = 2$, the rankings of principles in total investment are $EL > PRO > EA$. This proven finding will be further illustrated with computations.

In Figure 1, both agents have equal starting wealths, $w_1 = w_2 = 3$. Graph A shows what happens to the s_1^* when w_1 continuously increases from 3 to 6, while w_2 is equal to 3. The computation with given parameters at the beginning of the section is reflecting an intuitive finding of our results. The level of wealth influences investment levels when all other parameters are held constant. Consistent with the idea behind DARA, wealth increase has a positive effect on investment, and s_i increases under all principles.

Graph B shows reactions of s_1^* to the changes in w_2. We consider the case where w_2 continuously increases from 3 to 6 this time, and w_1 is equal to 3. This graph aims to clarify the influence of the increase in w_2 on s_1^*. By its nature, under PRO, s_1^* is not affected by changes in w_2. As a result, the black line is flat, and s_1^* is constant through the levels of w_2. Figure 2 illustrates the computations in further detail by drawing each of the graphs on a separate plane.

EA is the rule in favor of smaller shareholders. Note that s_i^{EA} starts from the same value in both Graph A and B when $w_i = w_j = 3$. We saw that in Graph A, investment rises as own wealth does so. This means 1 will invest more when w_1 increases. So 1 starts to be a bigger shareholder, and we know that EA is in favor of smaller shareholders in the state of bankruptcy. Even though she gets a disadvantage by being a bigger shareholder, the effect coming from her own wealth overrides the disadvantage of holding more shares. In Graph B, on the other hand, w_1 does not change, and there is no own wealth effect on s_1^*. However, as s_2^* increases thanks to w_2, 2

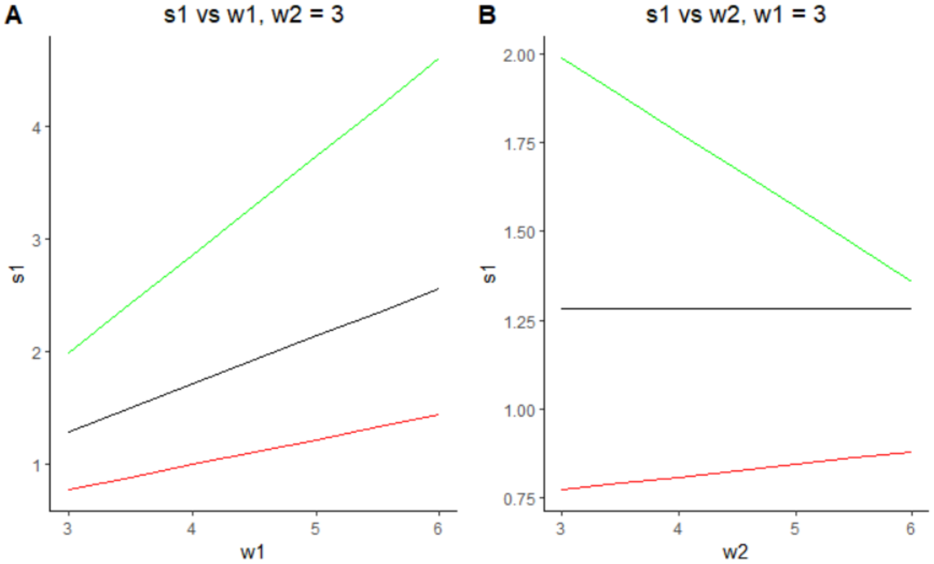

Fig. 1 Green line: EL, Black line: PRO, Red line: EA.

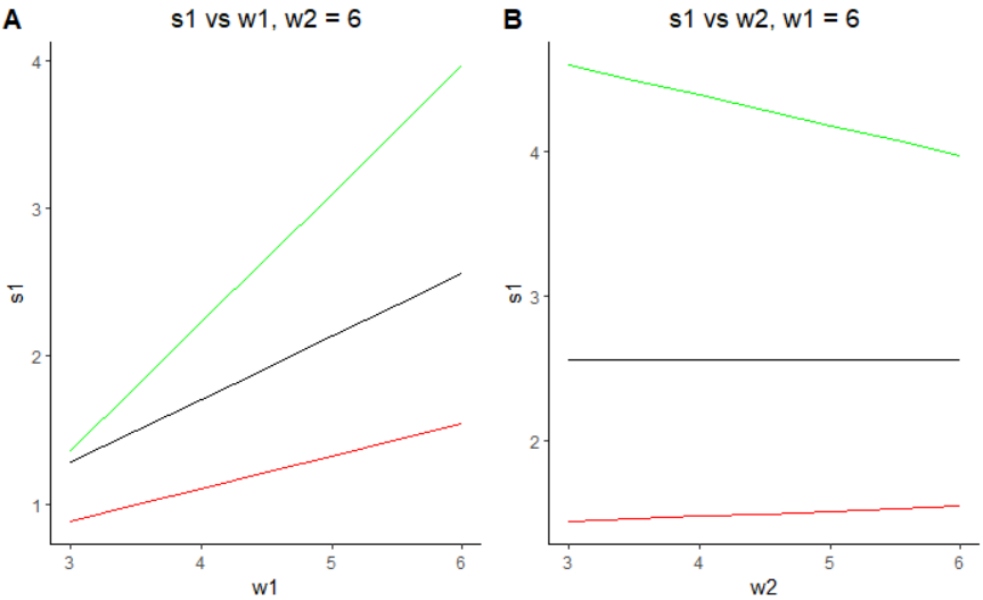

Fig. 2 Green line: EL, Black line: PRO, Red line: EA.

becomes the bigger shareholder. Now the situation of being a bigger shareholder is less likely for 1, so s_1^* slightly increases. This is important to see, even though w_1 does not change, s_1 changes related to the change in w_2. We can see the red line in B is flatter than the one in A.

EL is the rule in favor of those who invested more in the project. In that sense, in Graph A, EL has the greatest slope. When w_1 becomes greater than $w_2 = 3$, 1 starts to hold more shares than 2. So she gets the advantage of sharing the loss equally. That is the reason behind the green line is steeper than PRO. If the project goes under, 1 will not bear the loss of the whole amount she invested but share it equally with 2.

In Graph B, since w_2 rises, the explanations above apply to her investment attitude. As a result, 1 starts being holding less share since w_2 and s_2^* went up. Finally, her reaction to an increase in the opponent's investment will be decreasing her investment gradually. Since w_i is constant, there is no wealth effect thanks to an increase in w_1 like Graph A. What we see is the effect coming from the opponent's wealth, w_2. We can see the green line in both graphs starts at the same level of investment.

In Figure 2, initial wealth of the investors are not equal. In graph A 2 has greater wealth than 1 and in B it is the opposite. Graph A shows what happens to the s_1^* when w_1 goes from 3 to 6, while w_2 stays constant at 6. Similar to Figure 1, Graph A in Figure 2 also reveals that the change in w_1 has a positive effect on s_1^*. In Graph

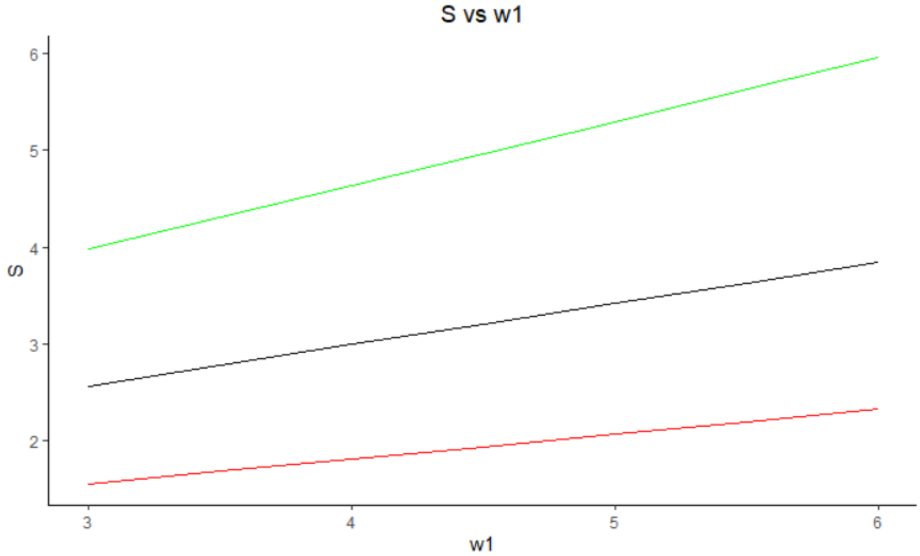

Fig. 3 Green line: EL, Black line: PRO, Red line: EA.

B, 1 has more wealth this time, and w_2 reaches to her wealth. The whole process of PRO is the same as what happened in Figure 1.

Under EA, in Graph A, w_1 starts at 3, and this means 1 is more likely to be in an advantageous situation by holding less share than 2. Nevertheless, the rise of w_1 results in an increase in investment. In B, 1 is a relatively rich one in this pair, and while the wealth difference decreases, 1 raises her investment. While the wealth difference is decreasing, also the difference of s_1^* and s_2^* decreases. For every value of w_2 while it increases, 1 faces less punishment from being a leading shareholder. That creates a positive reflection of s_1^* and yields a slight increase.

Under EL, in Graph A, since 1 is the poorer agent, we can say 2 invests more when $w_1 = 3$. So, 2 will suffer from the equal division of losses. The green line is steeper than the other two lines because both own wealth increase effect and lowering the wealth difference as a secondary effect has a positive influence on s_1^*. In B, 1 has the advantage of being relatively rich and is investing more than she would in a case of equal wealth. As w_2 rises and the wealth difference disappears, 1 gradually loses the advantage of EL and decreases s_1^*. Figure 4 will show the computations in detail.

Graph A in both Figures 1 and 3 shows the effect of personal wealth increase. Aforementioned, wealth change is not necessary to be actualized. A comparison of two agents with different wealth would also be enough, and our findings continue to apply.

Let us say there are two people with different decreasing absolute risk aversion (trying to make an inference for real life). And they have some value of wealth in the beginning. If we raise their wealth, we will observe the impact of their own wealth increase under all three principles. Since under PRO, w_1 is the only wealth component of s_1^*, we can see the effect of agent's own wealth clearly. Under the other two principles, the result would consist of the combination of own wealth's and wealth of others' effects.

Graph B in both Figures 1 and 3 shows the effect of an increase in the opponent's wealth. In the analysis part, thanks to the equal risk aversion assumption with γs, we can be sure that if an agent has a greater wealth than the other, she would invest more than the other. Nevertheless, when people who have different risk aversion for the same amount of wealth get involved in the investment, even if one has greater wealth, she might not make more investment than others. However, this does not lead the finding to lose its experimental interest. It can still be contested in an experimental study.

Total investment under the three principles is shown in Figure 3. It clearly shows the ranking between principles in terms of S is $EL > PRO > EA$. With the parameters of the model specified in the beginning of this section, EL has the steepest slope. Wealth difference gives the greatest rise to total investment under EL. Another fact is the one's own wealth increase overrides the effect of opponent's wealth increase when they encounter.

6 Conclusion

We study a bankruptcy problem with $n \geq 2$ agents endowed with DARA utility functions, where we focus on equilibrium properties of three bankruptcy rules, namely, PRO, EA, and EL. Our first set of results, proposition 1, 2, and 3 establish the equilibrium behavior of agents for the bankruptcy corresponding to PRO, $AP[\alpha]$, and

$LP[\alpha]$. These three propositions are stated and proved for the general case of $n \geq 2$ and possibly differing degrees of absolute risk aversions.

Due to the complex nature of the equilibria in the general version, we switch to a two-agent case (with a shared risk aversion parameter for both agents) in further analysis of the equilibria.

Our first set of results from further analysis concerns the effect of own wealth and other's wealth on the equilibrium investment. It turns out that independent of the bankruptcy rule to be applied in case of the project failing, an increase in own wealth leads to an increase in equilibrium investment. An increase in other agent's wealth, on the other hand, leads to no change (resp. increase, decrease) if PRO (resp. EA, EL) is the bankruptcy rule applied.

We then turn to a comparison of equilibrium total investment levels for different bankruptcy rules. We show that in terms of S, $EL > PRO > EA$. Our last section provides several illustrations from computations with different sets of parameters that summarize and hopefully further clarify our results.

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8 Appendix

Proof ***(Proposition 1)*** For each agent $i \in N$, $PRO_i(s) = \beta s_i$.

Return in case of success: $(w_i - s_i) + (1 + r)s_i) = w_i + rs_i$

Return in case of bankruptcy: $(w_i - s_i) + \beta s_i = w_i + (1 - \beta)s_i$

$U_i^{PRO}(s) = p\frac{1-\gamma_i}{\gamma_i}\left(\frac{w_i+rs_i}{1-\gamma_i}\right)^{\gamma_i} + (1-p)\frac{1-\gamma_i}{\gamma_i}\left(\frac{w_i-(1-\beta)s_i}{1-\gamma_i}\right)^{\gamma_i}$

by applying unconstrained maximization, we get:

$pr\left(\frac{w_i+rs_i}{1-\gamma_i}\right)^{\gamma_i-1} = (1-p)(1-\beta)\left(\frac{w_i-(1-\beta)s_i}{1-\gamma_i}\right)^{\gamma_i-1}$

$\frac{pr}{(1-p)(1-\beta)} = \left(\frac{w_i-(1-\beta)s_i}{w_i+rs_i}\right)^{\gamma_i-1}$

Let $A_i = \left[\frac{pr}{(1-p)(1-\beta)}\right]^{\frac{1}{\gamma_i-1}}$

Then $A_i = \frac{w_i-(1-\beta)s_i}{w_i+rs_i}$
$BR_i(s_{-i}) = \frac{[1-A_i](w_i)}{(A_ir+(1-\beta))}$
The above expression is achieved by first-order conditions.
$BR_i(s_{-i}) = \frac{\left(1-\left[\frac{pr}{(1-p)(1-\beta)}\right]^{\frac{1}{\gamma_i-1}}\right)(w_i)}{\left[\frac{pr}{(1-p)(1-\beta)}\right]^{\frac{1}{\gamma_i-1}}r+(1-\beta)}$
The denominator of the equilibrium investment is positive where $A_i > 0$. w_i , r and $(1-\beta)$ are already positive.

So the last expression for s_i to be positive is $1 - A_i > 0$. This can be transformed to $1 > \left[\frac{pr}{(1-p)(1-\beta)}\right]^{\frac{1}{\gamma_i-1}}$. Since in our model $\gamma_i < 1$, the power of the right hand side is smaller than 1. The inequality reduces to $pr > (1-p)(1-\beta)$.□

Proof ***(Proposition 2)*** PRO is the same principle as specified above. For each agent $i \in N$, $PRO_i(s) = \beta s_i$. Equal Awards (EA) can be described as a principle, dividing the survived amount of money equally amongst agents. For each agent $i \in N$, $EA_i(s) = \frac{\beta}{n}\sum_N s_i$ in case of bankruptcy.

Now for $AP[\alpha]$, when dividing the bankrupt value, proportionality rule is weighted with α, and the Equal Awards principle is weighted with $(1-\alpha)$. The way of dividing the amount in case of success remains the same, pure PRO.

Return in case of success: $(w_i - s_i) + (1+r)s_i = w_i + rs_i$
Return in case of bankruptcy: $(w_i - s_i) + \alpha\beta s_i + (1-\alpha)\frac{\beta}{n}\sum_N s_i$
$= w_i + \frac{(\beta+(n-1)\alpha\beta-n)}{n}s_i + (1-\alpha)\frac{\beta}{n}\sum_{N-i} s_j$
$U_i^{AP[\alpha]}(s) = p\frac{1-\gamma_i}{\gamma_i}\left(\frac{w_i+rs_i}{1-\gamma_i}\right)^{\gamma_i} + (1-p)\frac{1-\gamma_i}{\gamma_i}\left(\frac{w_i+\frac{(\beta+(n-1)\alpha\beta-n)}{n}s_i+(1-\alpha)\frac{\beta}{n}\sum_{N-i}s_j}{1-\gamma_i}\right)^{\gamma_i}$
by applying unconstrained maximization, we get:
$pr\left(\frac{w_i+rs_i}{1-\gamma_i}\right)^{\gamma_i-1} = (1-p)\left(\frac{n-\beta-(n-1)\alpha\beta}{n}\right)\left(\frac{w_i+\frac{(\beta+(n-1)\alpha\beta-n)}{n}s_i+(1-\alpha)\frac{\beta}{n}\sum_{N-i}s_j}{1-\gamma_i}\right)^{\gamma_i-1}$
$\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)} = \left(\frac{w_i+\frac{(\beta+(n-1)\alpha\beta-n)}{n}s_i+(1-\alpha)\frac{\beta}{n}\sum_{N-i}s_j}{w_i+rs_i}\right)^{\gamma_i-1}$
Let $A_i = \left[\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right]^{\frac{1}{\gamma_i-1}}$
Then $A_i = \left(\frac{w_i+\frac{(\beta+(n-1)\alpha\beta-n)}{n}s_i+(1-\alpha)\frac{\beta}{n}\sum_{N-i}s_j}{w_i+rs_i}\right)$
$BR_i(s_{-i}) = \frac{(1-A_i)(w_i)+(1-\alpha)\frac{\beta}{n}\sum_{N-i}s_j}{\left(A_ir+\frac{n-\beta-(n-1)\alpha\beta}{n}\right)}$
Let $B_i = (1-A_i)(w_i)$,
$C = (1-\alpha)\frac{\beta}{n}$,
$D_i = A_ir + \left(\frac{n-\beta-(n-1)\alpha\beta}{n}\right)$
$BR_i(s_{-i}) = \frac{B_i+C(S-s_i)}{D_i}$ where $S = \sum_N s_i$
Let $F_i = D_i + C$,
$BR_i(s_{-i}) = \frac{B_i+CS}{F_i}$
Solving this will give us:
$s_1^* = \frac{B_1+CS}{F_1}$
$+$ . .
$+$ . .
$+$ $s_n^* = \frac{B_n+CS}{F_n}$ =
$S = \frac{\left(\prod_{N-i}F_j\right)(B_i+CS)+\ldots+\left(\prod_{N-n}F_j\right)(B_n+CS)}{\prod_N F_i}$
$S = \frac{\sum_N\left[B_i\prod_{N-i}F_j\right]}{\prod_N F_i - C\sum_N\left[\prod_{N-i}F_j\right]}$ and by s_i^*'s formula, we have
$s_i^* = \frac{B_i+CS}{F_i}$.
The next step is replacing S inside the s_i^*. And the s_i^* becomes:
$s_i^* = \frac{B_i\left(\prod_N F_i - C\sum_N\left[\prod_{N-i}F_j\right]\right)+C\sum_N\left[B_i\prod_{N-i}F_j\right]}{F_i\left(\prod_N F_i - C\sum_N\left[\prod_{N-i}F_j\right]\right)}$

The expression which appears at the end of this process is the unique solution to the system $\{BR_i(s_{-i}) = s_i | i \in N\}$:
$$s_i^* = \frac{(1-A_i)(w_i)\left(\prod_N(A_ir+(1-\alpha\beta))-C\sum_N\left[\prod_{N_{-i}}(A_ir+(1-\alpha\beta))\right]\right)+C\sum_N\left[(1-A_i)(w_i)\prod_{N_{-i}}(A_ir+(1-\alpha\beta))\right]}{(A_ir+(1-\alpha\beta))\left(\prod_N(A_ir+(1-\alpha\beta))-C\sum_N\left[\prod_{N_{-i}}(A_ir+(1-\alpha\beta))\right]\right)}$$

Breaking down this expression, w_i is positive by definitions of w_i in this model. $\left(\prod_N (A_i r + (1-\alpha\beta)) - C\sum_N \left[\prod_{N\setminus i}(A_i r + (1-\alpha\beta))\right]\right) > 0$ has to be satisfied. $A_i = \left[\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right]^{\frac{1}{\gamma_i - 1}}$ is positive for any values of n, β, α, p, and γ_i defined in the model.

The last part in the nominator; $+C\sum_N \left[(1-A_i)(w_i)\prod_{N-i}(A_i r + (1-\alpha\beta))\right]$ is positive conditional on $1-A_i > 0$ since C, w_i, and $\prod_{N-i}(A_i r + (1-\alpha\beta))$ are already positive. Both expressions in the denominator have been examined before and stated as positive. Therefore, s_i^* being positive is conditional on the expression $1-A_i$ is positive or not.

$1 - A_i = 1 - \left[\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right]^{\frac{1}{\gamma_i - 1}}$

$1 - A_i > 0$ means $1 > A_i$ so,

$1 > \left[\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right]$

Since $\gamma_i < 1$ this makes the denominator of the power and the power negative. We can convert it to $1 > \left[\frac{(1-p)(n-\beta-(n-1)\alpha\beta)}{npr}\right]^{\frac{1}{1-\gamma_i}}$.

Now if we take $1-\gamma_i$ power of the both sides the inequality reduces to

$pr > (1-p)\frac{(n-\beta-(n-1)\alpha\beta)}{n}$.□

*Proof (**Proposition 3**)* Under EL, $EL_i(s) = s_i - \frac{(1-\beta)}{n}\sum_N s_i$.

$LP[\alpha]$ is a mixture of PRO and EL rules, with weights α and $(1-\alpha)$ respectively. $LP[\alpha]_i(s^*)$ denotes the exact return of investing s^*. The way of dividing the amount in case of success remains the same, pure PRO.

Return in case of success: $(w_i - s_i) + (1+r)s_i = w_i + rs_i$

Return in case of bankruptcy: $w_i - \alpha(1-\beta)s_i - (1-\alpha)\frac{(1-\beta)}{n}\sum_N s_i$

$= w_i + \frac{(\beta-1)(1+(n-1)\alpha)}{n}s_i - (1-\alpha)\frac{(1-\beta)}{n}\sum_{N_{-i}} s_j$

$U_i^{LP[\alpha]}(s) = p\frac{1-\gamma_i}{\gamma_i}(\frac{w_i+rs_i}{1-\gamma_i} + \eta_i)^{\gamma_i} + (1-p)\frac{1-\gamma_i}{\gamma_i}(\frac{w_i + \frac{(\beta-1)(1+(n-1)\alpha)}{n}s_i - \frac{(1-\alpha)(1-\beta)}{n}\sum_{N_{-i}} s_j}{1-\gamma_i} + \eta_i)^{\gamma_i}$

by applying unconstrained maximization, we get:

$pr\left(\frac{w_i+rs_i}{1-\gamma_i}\right)^{\gamma_i - 1} = (1-p)\left[\frac{(1-\beta)(1+(n-1)\alpha)}{n}\right]\left(\frac{w_i - \frac{(1-\beta)(1+(n-1)\alpha)}{n}s_i - \frac{(1-\alpha)(1-\beta)}{n}\sum_{N_{-i}} s_j}{1-\gamma_i}\right)^{\gamma_i - 1}$

$\frac{npr}{(1-p)(1-\beta)(1+(n-1)\alpha)} = \left(\frac{w_i - \frac{(1-\beta)(1+(n-1)\alpha)}{n}s_i - \frac{(1-\alpha)(1-\beta)}{n}\sum_{N_{-i}} s_j}{w_i + rs_i}\right)^{\gamma_i - 1}$

Let $A_i = \left[\frac{npr}{(1-p)(1-\beta)[1+(n-1)\alpha)]}\right]^{\frac{1}{\gamma_i - 1}}$

Then $A_i = \left(\frac{w_i - \frac{(1-\beta)(1+(n-1)\alpha)}{n}s_i - \frac{(1-\alpha)(1-\beta)}{n}\sum_{N_{-i}} s_j}{w_i + rs_i}\right)$

$BR_i(s_{-i}) = \frac{(1-A_i)(w_i) - \frac{(1-\alpha)(1-\beta)}{n}\sum_{N_{-i}} s_j}{\left(A_i r + \frac{(1-\beta)(1+(n-1)\alpha)}{n}\right)}$

Since $\frac{(1-\alpha)(1-\beta)}{n} > 0$, $w_i > 0$ and $(A_i r + \frac{(1-\beta)(1+(n-1)\alpha)}{n}) > 0$ by A_i being positive. So if $(1-A_i) < 0$ everyone's best response and optimal investment level would be equal to 0.

If $(1-A_i) > 0$, to a k amount of agents $(1, ..., k)$ might have best response functions $BR_i(s_{-i}) > 0$. $1-A_i$ being positive is examined in the previous rule too. But this time $(1-A_i) = 1 - \left[\frac{npr}{(1-p)(1-\beta)(1+(n-1)\alpha))}\right]^{\frac{1}{\gamma_i - 1}}$. As in the similar exercise in EA-PRO, $1-A_i > 0$ reduces to $pr > (1-p)\frac{(1-\beta)(1+(n-1)\alpha)}{n}$. We could explain the left-hand side as the return on unit investment when the firm succeeds, the right-hand side as the loss agents face in the case of bankruptcy.

Agents from $k+1$ to n will face $s_i^* = 0$ as an optimal investment level, because the 2^{nd} term in the nominator is subtracted from the first term and for some agents this makes the investment level negative.This order assumption of best response functions, $b_1 \geq ... \geq b_k > 0 = b_{k+1} = ... = b_n$, can be done thanks to the assumption of $\gamma_1 \geq ... \geq \gamma_n$.

Let $B_i = (1-A_i)(w_i)$, $C = \frac{(1-\alpha)(1-\beta)}{n}$, $D_i = A_i r + \left(\frac{(1-\beta)(1+(n-1)\alpha)}{n}\right)$, $BR_i(s_{-i}) = \frac{B_i - C(S-s_i)}{D_i}$ where $S = \sum_N s_i$.

Let $F_i = D_i - C$, $BR_i(s_{-i}) = \frac{B_i - CS}{F_i}$

Solving this will give us:

$s_1^* = \frac{B_1 - CS}{F_1}$

$+ \quad . \qquad\qquad .$

$+ \quad . \qquad\qquad .$

$+ \quad s_k^* = \frac{B_k - CS}{F_k} =$

$S = \frac{(\prod_{\{1,\dots,k\}\setminus\{i\}} F_j)(B_i - CS) + \dots + (\prod_{\{1,\dots,k\}\setminus\{i\}} F_j)(B_n - CS)}{\prod_k F_i}$

$S = \frac{\sum_k \left(B_i \prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right)}{\prod_k F_i + C\sum_k \left(\prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right)}$ and by s_i^*'s formula, we have
$s_i^* = \frac{B_i - CS}{F_i}$.
The next step is replacing S inside the s_i^*. And the s_i^* becomes:
$s_i^* = \frac{B_i\left(\prod_k F_i + C\sum_k\left[\prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right]\right) - C\sum_k\left[B_i \prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right]}{F_i\left[\prod_k F_i + C\sum_k\left(\prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right)\right]}$

The expression which appears at the end of this process is the unique solution to the system $\{BR_i(s_{-i}) = s_i | i \in N\}$:
$s_i^* = \frac{(1-A_i)(w_i)\left[\prod_k (A_i r + (1-\beta)\alpha) + C\sum_k\left(\prod_{\{1,\ldots,k\}\setminus\{i\}} (A_i r + (1-\beta)\alpha)\right)\right] - C\sum_k\left[(1-A_i)(w_i + (1-\gamma_i)\eta_i)\prod_{\{1,\ldots,k\}\setminus\{i\}} (A_i r + (1-\beta)\alpha)\right]}{(A_i r + (1-\beta)\alpha)\left[\prod_k (A_i r + (1-\beta)\alpha) + C\sum_k\left(\prod_{\{1,\ldots,k\}\setminus\{i\}} (A_i r + (1-\beta)\alpha)\right)\right]}$

Let us remember that this means the optimal investment level is positive for up to k agents and from k+1 to n, the optimal investment is 0. Breaking down this expression, w_i and the denominator term are positive. Also we are already examining the case where $1 - A_i > 0$. So for those k agents,

$B_i\left(\prod_k F_i + C\sum_k\left[\prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right]\right) > C\sum_k\left[B_i \prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right]$
is the condition for $LP[\alpha]_i(s^*) \geq 0$ and consequently $s_i^* > 0$.
For $\{k+1, \ldots, n\}$, $B_i\left(\prod_k F_i + C\sum_k\left[\prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right]\right) \leq C\sum_k\left[B_i \prod_{\{1,\ldots,k\}\setminus\{i\}} F_j\right]$ is the situation.
Carrying forward, the condition for $k+1$ agents would be
$B_i\left(\prod_{k+1} F_i + C\sum_{k+1}\left[\prod_{\{1,\ldots,k+1\}\setminus\{i\}} F_j\right]\right) > C\sum_{k+1}\left[B_i \prod_{\{1,\ldots,k+1\}\setminus\{i\}} F_j\right]$.
Thus when $k = n$, the unique Nash equilibrium is $s^* = (s_1^*, \ldots, s_n^*) > 0$ under $(1 - A_i) > 0$ and
$B_i\left(\prod_N F_i + C\sum_N\left[\prod_{N_{-i}} F_j\right]\right) > C\sum_N\left[B_i \prod_{N_{-i}} F_j\right]$.
The solution is:
$s_i^* = \frac{(1-A_i)(w_i)\left(\prod_N (A_i r + (1-\beta)\alpha) + C\sum_N\left[\prod_{N_{-i}} (A_i r + (1-\beta)\alpha)\right]\right) - C\sum_N\left[(1-A_i)(w_i)\left(\prod_{N_{-i}} (A_i r + (1-\beta)\alpha)\right)\right]}{(A_i r + (1-\beta)\alpha)\left(\prod_N (A_i r + (1-\beta)\alpha) + C\sum_N\left[\prod_{N_{-i}} (A_i r + (1-\beta)\alpha)\right]\right)}$.□