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Managing Reputation Incentives

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Abstract

In a repeated interaction, one tries to maximize his/her current payoffs and considers the future implications of the decision. Reputation concerns arise when there is incomplete information regarding the preferences of the counterparts. I analyze the finitely repeated interaction between an agent and a principal where either the agent or the principal sets the allocation of the relative stakes. The agent prefers to start the career path with larger stakes, decreasing the stakes gradually. By starting large, the (good) agent creates an environment where the future interaction becomes less valuable. Hence the incentive to take the right action in the current period prevails reputation concerns. On the other hand, the principal allocates the stakes so that the interaction starts small. By doing so, the principal benefits from the reputation concerns of the agent. The principal updates his belief on the agent through interaction - stakes increase as the agent's reputation rises. The findings highlight the optimal design of a career path under different conditions.

Keywords: Reputation, Repeated Games, Gradualism.

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1 Introduction

I analyze a repeated interaction between an uninformed principal and an informed and potentially bad agent. The principal can be described as an employer who does not have the necessary information. He needs an informed agent to delegate the authority for a project that will last for a finite number of periods. In each period of the relationship, a state of the world is realized. While the principal and the good agent favor the action that matches the state of the world, the bad agent prefers the higher action in the stage game.

In an influential paper, Ely and Välimäki (2003) (hereinafter EV) showed that reputational concerns might sometimes hurt an agent, a situation that they call "bad reputation". More precisely, they analyzed an interaction between a short-term principal and a long-term agent in an infinitely-repeated setup. They showed that if there is a positive probability of the agent being a bad type, then the principal does not hire the agent in the equilibrium. The reasoning is as follows. The good agent has a short-term incentive to take the correct action and a long-term incentive to increase her reputation. There are infinitely many periods to be played, hence the reputational incentives of the agent triumph. If there is a positive probability that the agent is a bad type, then the agent prefers to reveal herself as good to future employers. Thus, she chooses the action that will increase her reputation rather than the necessary action. The agent is not hired in all renegotiation-proof Nash equilibria, leading to the loss of all surplus.

Infinite periods analysis seems more suitable in interaction with no predefined number of periods. However, in numerous real-world applications, the interaction length is somewhat known. Consider, for example, a public project regarding social welfare, which is to be finalized in a certain amount of time. The main difference of the finitely-repeated interaction from the infinitely-repeated one is that reputation concerns are expected to decrease through time. Hence, I have a different sort of reputation concern in force. I start with the analysis of the interaction with equal stakes. [1] It corresponds to the classical analysis where each period's decision has the same weight, and there is no discounting factor. I show the existence of a period in which the agent is hired if her reputation as a good type is sufficiently high. If the interaction lasts for a predefined number of periods, there is a period after which there are no reputation concerns.

In the equal-stakes model, the principal does not hire the agent if her reputation as a good type is sufficiently low. I introduce endogenous stakes to the analysis and ask the following questions: Suppose that the agent or the principal can choose the size of the stakes involved in their relationship in every stage of the game. Does the equilibrium payoffs improve with the introduction

[1] It is a finitely-repeated application of Ely and Välimäki (2003) where both players are long-term.

of endogenous stakes? Does the principal hire the agent for lower reputation levels (compared to equal stakes)? Would they choose the stakes to prevent bad reputation, i.e., let the good and the bad agents separate at the beginning of the game? Who would benefit from the reputation concerns? What is the equilibrium path of the size of the stakes? Does the principal or the agent start the relationship large or small?

Since the bad reputation may hurt the good agent and the principal, one would think that they would choose the stakes to eliminate reputational incentives. It can be achieved by starting the relationship large and assigning smaller stakes to the future. The less important the future is, the fewer reputation concerns are there. This intuition is correct in specific environments. In particular, the agent prefers to start large. When the agent chooses the stakes, she puts high stakes to initial periods. Hence, small stakes are left to the future periods, and reputation concerns are prevented. Why would the agent prefer to start large? There is no incomplete information about the principal's preferences. It is optimal to set stakes as high as possible in any period the agent expects to be hired.

However, I show that this intuition does not work in some instances. One critical case is as follows. Suppose a principal faces an agent that is a strategic bad type with sufficiently high probability. In that case, he chooses to start the relationship small and gradually increases the size of the stakes. Furthermore, the equilibrium behavior induced by this choice of stakes involves a bad reputation. The good agent always plays the action that contributes to her reputation while the bad agent completely mixes until the last two periods. This result is surprising because this is also the equilibrium most preferred by the principal, i.e., he chooses to use a bad reputation to his advantage if the bad agent is strategic and the initial reputation is bad enough. Moreover, I show that the principal does not start large in any equilibrium.

There are several empirically relevant scenarios, some of which have already been discussed by EV, that fit one of the above environments. Lawyers might need a large case to kick start their careers. If a lawyer decides to start with minor cases, the client may believe that the lawyer will be more interested in increasing her reputation rather than the case. The lawyer can signal the potential client that she will not have reputational incentives by starting with large cases. On the other hand, a government official appointing judges (or attorney general assigning cases to state attorneys) may prefer to start with small cases. Having smaller cases in the initial periods minimizes the harms of the reputational incentives. Our analysis allows us to make equilibrium predictions in any such environment. I need to note that our results should be interpreted carefully since I am not considering other possible determinants of the allocation of stakes over time, such as learning about the agent's ability, learning on the job, etc.

I show that if the agent sets stakes, she prefers to start the interaction large and decrease the stakes gradually. There are instances when the agent holds no reputation concern in the equal-stakes model. For example, the interaction length may be short, or the value obtained from the project may be small. In that case, the agent's ability to design the career path benefits her in two aspects. First, she obtains a higher payoff by allocating the high stakes to the periods where she is hired for sure. Second, she is also hired for lower reputation levels (in which she is not hired in the equal-stakes model). In the complementary case, the agent has reputation concerns in the equal-stakes model (It occurs if the interaction lasts for many periods or the value of the project is large). By starting large, the agent leaves sufficiently small stakes to the future interaction so that she has no incentive to invest in her reputation. Therefore, the "bad reputation" outcome can be avoided by changing the allocation of relative stakes. The agent obtains a higher payoff if she is hired.

However, the agent's ability to set the stakes does not always benefit her. Consider the following case. The agent has a small initial reputation level and holds reputation concerns in the equal-stakes model. The principal expects a non-negative payoff, hence, hires the agent. However, if the agent sets the relative stakes, the principal may not hire the agent. The agent can not commit to a particular allocation of stakes for all periods at the beginning of the interaction. Once the agent's reputation level increases above a certain level, the agent will put the highest possible stakes in every successive period. Mostly, it corresponds to the allocation that yields a zero continuation payoff to the principal. As the principal gets a negative payoff in the earlier periods with reputation incentives, it means a negative continuation payoff for the principal in the first period. Hence, the principal never hires the agent.

When the principal sets relative stakes, he never prefers to start large. Interestingly, the principal never sets the stakes such that the agent would strictly prefer to play stage-game maximizing action. The principal sets the stakes so that at least one type of agent is indifferent between playing both actions. Endogenous stakes improve the principal's payoff, and the agent is hired for lower reputation levels. Additionally, I show that if the agent's initial reputation is sufficiently bad, the principal starts small and increases stakes gradually. The bad agent is indifferent between the two actions in each period. The good agent's type is not revealed until the last period in the equilibrium. The principal sets stakes for each period so that he can manipulate the reputation incentives of the agent and get the maximum expected value from the project. It is consistent with the reputation literature on gradualism.[2] Compared to the setup where each period receives equal weight, the principal's ability to set the relative stakes enables him to hire the agent with lower initial reputation levels.

[2]Watson (1999, 2002), Andreoni and Samuelson (2006) are the most prominent ones.

2 Literature Review

A vast literature exists on the role of reputation in two-agent or principal-agent problems. Papers by Kreps and Wilson (1982) and Milgrom and Roberts (1982) were the first to formalize the effects of the reputation that pioneered the conventional wisdom that the reputation concern increases commitment power. These papers analyzed models in which a long-term player meets with several short-term agents, and they showed that adding even some imperfect information leads to the rise in the reputation effect. An incumbent monopolist faces short-term entrants. If there is a small probability that the incumbent monopolist is tough, he may want to maintain a reputation for toughness to defer the other agents' entry. Kreps and Wilson (1982) and Milgrom and Roberts (1982) showed that the introduction of the repeated actions and asymmetry in information leads to a reputation effect. In that framework, even if the predatory action is irrational for the monopolist in the stage game, he takes that action to gain a reputation as a predator. This action helps the monopolist to prevent short-term players from entering the market. He enjoys his monopolistic power in the future by investing his reputation today.

Fudenberg and Levine (1989, 1992) applied these results to a more general set of games where a long-run player faces short-run opponents. Schmidt (1993) and Celetani et al. (1996) analyzed the role of reputation when there are two long-run players. Schmidt concludes that if player 1 is sufficiently more patient than the second player, the reputation effect works on the first player's advantage. Likewise, Celetani et al. (1996) showed that if a player (whose type is not known by the other player) is more patient than his opponent, he can exploit the less patient player's uncertainty by holding a reputation as a commitment type. This literature following Kreps and Wilson (1982) and Milgrom and Roberts (1982) suggested that reputation building is an effective tool to increase payoff compared to an environment with no reputation incentives.

Holmström (1999) defined the reputation incentive as an employee's career motives and showed that the reputation concerns could be beneficial or harmful depending on the level of conflict (or alignment) between the employer and the employee. Morris (2001) analyzed a twice repeated interaction in a cheap-talk environment with the noise. He defines three effects of the reputation incentives. Firstly, the "discipline effect" of the reputation concerns leads the bad agent to choose an action that hurts the principal less. In our set-up, this effect leads to bad agent playing the action that matches the state of the world, rather than always playing the higher action. Secondly, through the "sorting effect", the principal can update his belief about the agent through the relation. The first two effects of the reputation concerns improve the equilibrium payoffs. However, the third effect, "political correctness" leads the good agent to choose the action that will increase her reputation, rather than choosing the right action. All in all, Morris argues that the reputation concerns' overall effect is unambiguous.

Ely and Välimäki (2003) continued to show that the aforementioned losses from a bad reputation could be avoided if the principal is also a long-term player. If both the principal and the agent are long-term players, the agent goes through an evaluation phase. In the evaluation phase, the agent is hired irrespective of the action observer. If the reputation of the agent gets above some level, then she is hired for some periods. I show that the equilibrium behavior is similar in a finitely-repeated interaction. I start with the EV model to define stage-game payoffs. I focus on the interaction between long-term principal and agent. The first point distinguishing the current paper is the span of the interaction - I look at a finitely-repeated interaction. More importantly, I introduce endogenous stakes to the analysis. Either the agent or the principal sets the stakes. This enables us to examine who benefits from the reputation incentives.

Ely et al. (2009) defined a more general setting to understand when a game is a "bad reputation" game. They had a richer type set in this setting. They showed that if the probability of an "unfriendly" commitment type is sufficiently high and the long-run player is patient enough, then the utility of the long-run player converges to exit payoff in any equilibrium. Then, the effect of the reputation is "bad". Unfriendly commitment types are the agents who commit to bad actions. Moreover, Ely et al. (2009) showed that if there is a Stackelberg type likely enough, the bad reputation result does not hold.

Our paper partially belongs to the literature on gradualism. I show that the principal prefers gradualism to manage the reputation incentives of the agent. Across the literature on repeated interactions, it is widely accepted that starting the interaction with small stakes and increasing them over time leads to maximum payoffs. The pioneering papers on gradualism are those by Watson (1999, 2002). He studied infinitely repeated prisoners' dilemma game where the partners jointly and dynamically determine the periods' relative weights. There is two-sided incomplete information about the types of players. A low type prefers to betray, whereas a high type chooses to cooperate as long as the other player cooperates. Both of Watson's papers conclude that if the interaction starts small, then cooperation is viable, and there is an equilibrium where high types cooperate perpetually.

Another paper on gradualism is by Andreoni and Samuelson (2006). They experimented with a twice-played prisoner's dilemma game. They varied relative stakes of the periods and found that the stakes' optimum allocation is approximately one-third to two-thirds. They experimentally verified that starting small is the best concerning the total payoff of the players. Andreoni et al. (2019) extended that analysis by endogenizing the relative stakes' decision. In their experiments, players chose the relative stakes, and they found that starting small increases the cooperation and social payoff. Grosskopf and Sarin (2010) provided another experimental work where a long-run player faced short-run players. They found that the reputation is seldom harmful, and its beneficiary effects are not tremendous as thought.

Atakan et al. (2020) studied a repeated interaction between a sender and a receiver for a finite number of periods. In their model, the sender is informed and interacts with an uninformed and possibly bad receiver. The sender holds information and decides whether or not to share accurate information with the receiver in each period. The sender or the receiver sets the importance parameter in each period in this framework. With its finite nature and dynamically set relative stakes, Atakan et al. (2020) provide another starting point for our paper. They showed that it is payoff-dominant for both receiver and the sender to start the interaction small and increase gradually through time.

3 The model

I will first define the basic model where the stakes are equal across periods. I do not have a discounting factor. Ely and Välimäki (2003) provide the analysis with discounting close enough to 1. Hence, I assume a discounting factor equal to 1 to avoid future confusion when I introduce the stake variable later.

An informed agent meets an uninformed principal for n periods. Let N be the index set of the periods. In each period $i \in N$, the state $\theta_i \in \{0,1\}$ is realized. Each state is equally likely, i.e. $prob[\theta_i = 0] = prob[\theta_i = 1] = 1/2$ and independent across the periods. The agent observes the state, whereas the principal does not.

At the beginning of each period, the principal decides whether to hire the agent or not. The principal's period i action is defined by λ_i, where $\lambda_i = 0$ means he does not hire the agent and $\lambda_i = 1$ means the principal hires her. If the principal does not hire the agent, the interaction ends. If $\lambda_i = 1$, then the agent observes the true state and takes action $a_i \in \{0,1\}$. The payoffs are realized, and the game moves to the next period.

There are two types of agents: $\{g,b\}$ - *good* and *bad* types. The good agent shares the same preferences with the principal, while the bad agent favors action 1. The agent is privately informed about his type. $\rho_1 \in [0,1]$ measures the initial reputation of the agent being good.

For any $i \in N$, let H_i be the set of all histories before decision i is made, i.e. $H_i = (o_1, o_2, ..., o_{i-1})$ where $o_i = (\lambda_i, \alpha_i)$. $\alpha_i \in \{\emptyset, \{0,1\}\}$ denotes the outcome of the period i action. If the agent is not hired in period i, then $\alpha_i = \emptyset$. The strategy of the period i principal is given by $\Lambda_i : H_i \to \{0,1\}$ where $\Lambda_i(h)$ is the principal's choice of λ_i. Principal's belief about the type of agent is defined by the probability of agent being good in period i: $\rho_i : H_i \to [0,1]$. The principal observes all the previous actions of the agent but does not observe the true state in these periods. The good agent observes the state of world before taking action but the bad agent does not. The good and the bad agent moves after histories of this type: $I_i^g = (h, \lambda_i, \theta_i,)$ and $I_i^g = (h, \lambda_i)$, respectively, where I_i^a is the period i information set of the agent type $a \in \{g,b\}$. The good and the bad agent's (mixed) strategy is defined by the probability of agent playing action 0, given by $\mu_i(h,1,\theta_i)$ and $\nu_i(h,1)$, respectively. $\mu_i \& \nu_i : \mathfrak{I}_\mathrm{i} \to \Delta[0,1]$ where $\mathfrak{I}_i$ is the set of all period i information sets.

The timing of the stage game in period i can be summarized as follows:

1. The principal makes the hiring decision. If $\lambda_i = 0$, then the game ends. Both parties get a payoff of 0.
2. If $\lambda_i = 1$, nature chooses the state $\theta_i \in \{0,1\}$.
3. The (good) agent observes θ_i and takes the action $a_i \in \{0,1\}$

The stage game payoffs are given by Table 1.

Good agent (and principal)

	$\theta_i = 0$	$\theta_i = 1$
$a = 0$	k	$k-1$
$a = 1$	$k-1$	k

Bad agent

	$\theta_i = 0$	$\theta_i = 1$
$a = 0$	$k-1$	$k-1$
$a = 1$	k	k

Table 1: The Payoff Table

where $k \in (0, 1/2)$ is assumed to be the payoff accruing from the project in each period. This constraint on k is made for the following reason. In the one-shot game where there are no reputational incentives, the principal's stage game payoff is given by $\rho k + (1-\rho)(k - 1/2)$. If $k \geq 1/2$, then the principal hires the agent irrespective of the reputation level - even if the agent is a bad type for sure. On the other hand, negative k implies a negative project value. Hence, the principal never hires the agent, even if she is a good type for sure.

The expected continuation payoff of the player type $A \in \{g,b,p\}$ (good, bad agent and principal, respectively) in period i is given as follows: $V_i^A = \frac{1}{n} \sum_{j=1}^{n} U_j^A$ where U_i^A denotes the stage game payoff. $\frac{1}{n}$ is the weight of each period in the benchmark analysis.

Period i stage game strategy of the agents is summarized with the Table 2. $\mu_i(h,1,\theta_i)$ and $\nu_i(h,1)$ is denoted by $\mu_i^{\theta_i}$ and ν_i for the ease of exposition. $\mu_i^{\theta_i}$ denotes the probability with which the good agent plays action 0 given $\theta_i \in \{0,1\}$ and ν_i denotes the probability with which the bad agent plays action 0 in any state of the world. I assume that the bad agent does not observe the state of the world. She is the type that mixes between two actions irrespective of the state of the world.

	$\theta_i = 0$	$\theta_i = 1$
Good agent	0 with probability μ_i^0 1 with probability $1-\mu_i^0$	0 with probability μ_i^1 1 with probability $1-\mu_i^1$
Bad agent	0 with probability ν_i 1 with probability $1-\nu_i$	

Table 2: The Strategies Table

Given the above-defined strategies, the principal's belief about the type of the agent in period $i+1$ is updated by Bayes' rule as following:

$$\rho_{i+1}(a_i = 0|h) = \frac{\rho_i\left(\mu_i^0 + \mu_i^1\right)}{\rho_i\left(\mu_i^0 + \mu_i^1\right) + 2(1-\rho_i)\nu_i}$$

$$\rho_{i+1}(a_i = 1|h) = \frac{\rho_i\left(2 - \mu_i^0 - \mu_i^1\right)}{\rho_i\left(2 - \mu_i^0 - \mu_i^1\right) + 2(1-\rho_i)(1-\nu_i)}$$

I focus on the perfect Bayesian equilibria with Markovian property. In that regard, history matters only in terms of its effect on the reputation of the agent. Thus, if the agent has the same reputation level after two different histories, then both the agent's and the principal's action is the same in both cases. The decisions depend on the past observations only through their effect on the reputation of the agent.

Markovian Property. For any $i \in N$ and $h, h' \in H_i$: $\rho_i(h) = \rho_i(h')$ implies $\lambda_i(a_i|h) = \lambda_i(a_i|h')$, $\mu_i(h) = \mu_i(h')$ and $\nu_i(h) = \nu_i(h')$.

For notational simplicity, I will suppress h in the notation and denote the action of the principal after observing a_i given h by $\lambda_i(a_i)$, i.e. $\lambda_i(a_i) \equiv \lambda_i(a_i|h)$. In the same manner, $\rho_{i+1}(x) \equiv \rho_{i+1}(a_i = x|h)$ where $x \in \{0,1\}$.

When there are multiple equilibria, I use agent-optimality as the equilibrium selection criteria. The agent-optimal equilibrium is the equilibrium that leads to the highest expected continuation payoff for the agent among all equilibria.

Define reputation concern as taking an action a_i other than stage-game payoff maximizing action which lead to $\rho_{i+1}(a_i) > \rho_i$. In other words, if the agent has a reputation concern in period i, then she has an incentive to play the action that will contribute to her reputation level, even if that action hurts the stage-game payoffs. Playing such an action with a positive probability is only sequentially rational if the agent gets a sufficiently high payoff following that action. I show that reputation-driven action is action 0.

First, note that the agent's payoff is increasing in reputation level in any agent-optimal equilibrium. This condition may seem to be natural, but there are equilibria in which a lower reputation level leads to a higher payoff to the agent. Consider period i such that no reputation concern in period i onward is an equilibrium. In words, the good agent plays the action matching the state of the world, and the bad agent plays action 1. Now suppose there is an equilibrium in which for some $\rho_i^{'}$, both type of agent play action 1 with probability 1 if $\rho_i > \rho_i^{'}$. Both types of agents play the stage-game payoff maximizing action for lower reputation levels, i.e., no reputation concern. Given $\rho_i > \rho_i^{'}$, define the principal's off-the equilibrium path belief as $\rho_{i+1}(0) = 0$. This assessment may constitute an equilibrium. In that case, the agent may get a higher payoff if $\rho_i \leq \rho_i^{'}$ (compared to $\rho_i > \rho_i^{'}$). Hence, the agent does not always prefer a higher reputation level in *any* equilibrium. I show that a higher reputation level implies a higher payoff for the agent in the agent-optimal equilibrium.

Suppose for the contradiction that action 0 decreases the agent's reputation in period i. As the agent prefers a higher reputation level, the bad agent plays action 1 with probability 1 in such period. If good agent plays action 0 with a positive probability, then $\rho_{i+1}(0) = 1$ by Bayes's rule, contradicting $\rho_{i+1}(0) < \rho_i$. Hence, action 0 can only decrease the agent's reputation if it is played with 0 probability, i.e., not observed on the equilibrium path. There are such babbling equilibria where both types of agent play action 1 and the principal puts a sufficiently low belief on the probability of the agent being good type after action 0. No type of agent has an incentive to deviate to play action 0 if the future is sufficiently important. In such case, the agent takes the costly action 1 (when $\theta_i = 0$) and her reputation is not updated ($\rho_{i+1}(1) = \rho_i$). I show that there always exists a separating equilibrium which implies a higher payoff to the agent. The Lemmas 1, 2 and 3 formalize the argument.

Lemma 1. *In the agent-optimal perfect Bayesian equilibrium,*

$$\rho_{i+1}(a_i = 0|h) \geq \rho_i \geq \rho_{i+1}(a_i = 1|h) \ \ \forall i \text{ and } \forall h \in H_i$$

which holds strictly when $\rho_i \neq 1$.

Proof. Appendix. □

If $\theta_i = 0$, then the good agent will play action 0 with probability 1 ($\mu_i(h,1,0) = 1$). Note that the good agent has no incentive to act differently from the state observed when $\theta_i = 0$. Thus, playing action 1 with a positive probability when the state is 0 can not be an equilibrium strategy for the good agent. It is easy to see as playing action 1 hurts both the stage game payoff and the reputation. When the true state is 1, she has a reputational incentive to play 0 as it will increase her reputation as a good agent. Hence,

Lemma 2. *In the agent-optimal perfect Bayesian equilibrium,* $\mu_i(h,1,0) = 1 \ \ \forall i$ *and* $\forall h \in H_i$.

Proof. Appendix. □

Lemma 1 and Lemma 2 imply that v_i can not be above a certain level in any equilibrium. The reasoning is straightforward. If the bad agent plays action 0 with a sufficiently high probability compared to the good agent, then $\rho_{i+1}(1) > \rho_{i+1}(0)$ may be the case. But then, it implies that the agent has no motivation to play action 0. Thus, the agent will deviate to play action 1 with probability 1. Hence,

Lemma 3. *In the agent-optimal perfect Bayesian equilibrium,* $v_i \leq \dfrac{1+\mu_i}{2}$.

Proof. Appendix. □

Thus, I can interpret the strategies μ_i and v_i as measuring the reputation concerns of the two types of the agent. For ease of exposition, I will write μ_i to refer to $\mu_i(h,1,1)$ because $\mu_i(h,1,0) = 0$ in the agent-optimal equilibrium.

The reputation concerns depend on n. It corresponds to the length of the interaction. The higher the number of periods is left to be played in the future, the (weakly) more reputation concerns agent will have. EV analyzes infinitely repeated setup; hence there are always reputation concerns if there is a positive probability that the agent is bad. However, in the finitely repeated setup, there is always a period i after which there are no more reputation concerns. Consider a critical period i after which agent is hired if $a_i = 0$ is observed. There are $n-i$ periods left in the future. Suppose $(n-i)k+k-1 \geq k$ and $\mu_i = 0$ and $\nu_i = 0$. Note that $\rho_{i+1}(0) = 1$, and hence, both type of agent has an incentive to deviate to play action 0. Hence, $\mu_i = 0$ and $\nu_i = 0$ can not be equilibrium strategies. On the other hand, if $(n-i)k+k-1 < k$, then $\mu_i = 0$ and $\nu_i = 0$ in any equilibrium. As playing action 0 leads to a maximum of $(n-i)k+k-1$, the agent prefers to play the stage-game maximizing action, leading to a strictly higher payoff, k. Hence, $n < \frac{1}{k} + i$ is the sufficient condition for the agent to not have reputation concerns in period i. The following proposition formalizes the argument for the first period.

Proposition 1. *If $n < \frac{1}{k} + 1$, then the agent has no reputational incentive in any Perfect Bayesian equilibria in the equal-stakes model, i.e. $\mu_i \& \nu_i = 0 \, \forall i \in N$.*

Proof. Appendix. ☐

3.1 One Period Model

I first provide the equilibrium analysis of the one-period model, i.e., $n = 1$. When the agent and the principal meet only once, the principal's objective is to maximize his stage game payoff. Thus, any reputational incentive of the good agent is harmful to the principal. Moreover, even if there are no reputational incentives, the principal does not hire the agent if the agent's reputation is below a certain level, more precisely $1-2k$. For the higher reputation levels, the principal hires the agent only if he believes the good agent plays action 0 with sufficiently low probability when the true state is 1. In other words, if the good agent has a strict reputational incentive in a period ($\mu_i = 1$), then she will not be hired. It follows from the following stage-game payoff of the principal:

$$U^P = \frac{1}{2}\rho k + \frac{1}{2}\rho\mu(k-1) + \frac{1}{2}\rho(1-\mu)k + (1-\rho)\left(k - \frac{1}{2}\right)$$

A strategic bad agent's strategy does not depend on the state. Hence, it does not affect the principal's payoff in a one-shot game. On the other hand, the principal's payoff is decreasing in μ. Given $\rho \geq 1-2k$, U^P is non-negative if μ is below μ' such that:

$$\mu' = \frac{2k+\rho-1}{\rho}$$

Therefore, in a one-shot game, the agent is hired if and only if $\rho \geq 1-2k$ and $\mu \leq \mu'$. Regarding the agent's behavior in the one-shot game, it is clear that there will be no reputational incentives. Hence, $\mu \& \nu = 0$. Thus, the agent will be hired as long as the reputation is greater than or equal to $1-2k$.

4 Equal-stakes

Let $t \in \{1,2,...,n-1\}$ and define

$$R(t,n) = \frac{(1-2k)2^{t-1}}{(1-2k)2^{t-1} + (n-t+2)k}$$

Suppose the initial reputation is sufficiently high, i.e., $\rho_1 \geq R(n,n)$. There is an equilibrium where the agent has no reputation concerns and is hired every period irrespective of the history. It is the agent-optimal equilibrium because the agent gets the highest possible payoff. Consider period n. The lowest reputation level of the agent is attained if action 1 is observed in all the previous periods. Bayes' rule implies that if $\rho_1 \geq R(n,n)$ then $\rho_n \geq 1-2k$ after any history. Hence, the principal hires the agent in any period.

Next, consider middle initial reputation levels. The following lemma will be used in the further results.

Lemma 4. *Take any PBE and suppose that $\mu_i = \nu_i = 0$ for $i = j, j+1, \dots, n$. Then the agent is hired in period j if and only if $\rho_j \geq \frac{1-2k}{1+(n-j)k}$.*

Proof. Appendix. □

The principal hires the agent if the agent's reputation is above a certain level so that the principal expects a non-negative payoff. The principal's expected payoff depends on whether the agent will have reputation concerns in the remaining periods. Lemma 4 states the lowest reputation level for which the principal hires the agent if there will be no reputation concerns in the remaining periods.

Suppose the initial reputation is such that $R(t,n) \leq \rho_1 < R(t+1,n)$ for $t \in \{1,2,...,n-1\}$ and the agent has no reputation concern in the initial t periods i.e. $\mu_i = \nu_i = 0$ for $i = 1,2,\dots,t$. Bayes' rule implies, if no action 0 is played, then $\rho_t \geq \frac{1-2k}{1+(n-t)k}$ and $\rho_{t+1} < \frac{1-2k}{1+(n-t-1)k}$. By

Lemma 4, it implies the principal hires the agent in period t if he expects no reputation incentive. In addition, if action 1 is observed in period t, then the principal does not hire the agent in period $t+1$. In the equilibrium path where there is no reputation concerns in any period, $R(t,n) \leq \rho_1 < R(t+1,n)$ implies that the agent is hired for exactly t periods if no action 0 is observed. Naturally, if action 0 is observed in any period, then the agent is hired for the rest of the relationship.

So far, I have defined the equilibrium behaviors where there are no reputation concerns. Next, I will show the necessary and sufficient conditions for the absence of reputation concerns in any equilibrium. Define *critical period* i as a period in which $\lambda_{i+1}(0) = 1$ and $\lambda_{i+1}(1) = 0$. In words, if $a_i = 1$ is observed, then the agent is not hired for the remaining periods. On the other hand, if $a_i = 0$ is observed, she is hired for 1 or more periods in the future. Consider critical period t. Sequential rationality constraint of the agent implies that if $k > (n-t)k + k - 1$, then the agent has no incentive to play the costly reputation action. Now consider period $t-1$. Good and bad agents' sequential rationality constraints are as follows:

$$good: \; 2k + \frac{1}{2}(n-t)k > k - 1 + k + (n-t)k$$

$$bad: \; 2k > k - 1 + k + (n-t)k$$

Realize that the absence of reputation concerns in period t implies that there are no reputation concerns in period $t-1$. By induction, it implies no reputation concerns in any period $1,2,3...,t-1$. In addition, $n < \frac{1}{k} + t$ (by the sequential rationality constraint of the agent inf the critical period) implies there are no reputation concerns in period t onward. Moreover, $\rho_1 \geq R(t,n)$ implies that if the agent has no reputation concerns in the first t periods, then she survives the first t periods irrespective of the history. Thus, $\mu_i = 0$ and $\nu_i = 0 \; \forall i \in N$. Proposition 2 formalizes the argument.

Proposition 2. *In the agent-optimal perfect Bayesian equilibrium:*

1. *If $\rho_1 \geq R(n,n)$, then the agent has no reputational incentive, i.e., $\mu_i = \nu_i = 0 \forall i \in N$, and she is hired in every period irrespective of the history.*

2. *If there exists $t \in \{1,2,...,n-1\}$ such that $R(t,n) \leq \rho_1 < R(t+1,n)$ and $n \leq \frac{1}{k} + t$, then the agent has no reputational incentive, i.e., $\mu_i = \nu_i = 0 \forall i \in N$, and she is hired in periods 1,2,...,t after any history. In period $i \geq t+1$, she is hired if and only if action 0 is observed in some period $j < i$.*

3. *If $\rho_1 < R(1,n)$ and $n \leq \frac{1}{k} + 1$, then the agent has no reputational incentive, i.e., $\mu_i = \nu_i =$*

$0\,\forall i \in N$, *and she is never hired.*

Proof. Appendix. □

In Proposition 1, I stated that if $n < \frac{1}{k} + 1$, then there are no reputation concerns in any equilibrium. Proposition 2 states that, $n < \frac{1}{k} + 1$ is not necessary but sufficient condition for the absence of reputation concerns in the agent-optimal equilibrium. If the initial reputation of the agent is so that she will be hired for t periods (given no action 0 is observed), i.e. $R(t,n) \leq \rho_1 < R(t+1,n)$, then the condition becomes $n < \frac{1}{k} + t$. Given initial reputation level of the agent, if n is small or k small, then the agent has no reputation concern. Note that, the absence of reputation concerns implies that the bad agent always plays action 1.

Reputational incentives are determined by n and k: Larger k and n lead to stronger reputational incentives. On the other hand, if n or k is small, then both the good and the bad agents play their ideal actions. Therefore, there is no bad reputation in this case. Since the bad agent always plays action 1, if action 0 is observed in any period, the principal will hire the agent in every consecutive period. Whether the principal hires the agent in the first period depends on the initial reputation of the agent. If the reputation is large enough, i.e., $\rho_1 \geq R(1,n)$, the agent is hired in the first period. Next, I provide comparative statics and analyze how equilibrium behavior and payoffs vary with the parameters of the game as long as $n < \frac{1}{k} + t$ holds.

4.1 What happens when n increases?

If the length of the interaction increases, the threshold for hiring, i.e., $R(1,n)$ decreases. Mathematically, it is easily seen as $R(1,n)$ is decreasing in n. The larger the interaction lasts, the smaller stake each period receives. Consider an extreme scenario for illustration. The initial reputation is sufficiently small so that the agent is hired in the first period but is only hired in the second period if $a_1 = 0$ is observed. The agent is the bad type with sufficiently high probability; therefore, the principal's expected first-period payoff is negative. However, in the absence of reputation concerns, $\rho_2(0) = 1$. With a probability of $\frac{1}{2}\rho_1$, the principal gets an expected payoff of k in the remaining periods. In the equilibrium path, only uncertainty about the agent's type lies in period 1. Thus, the principal hires the agent for the lower reputation levels as the stake of the first period, i.e., $\frac{1}{n}$, decreases.

Next, consider t. As n increases, the number of periods the agent is hired (even if the history has no action 0) also increases. The intuition is as follows. Take an equilibrium in which there is no reputational concerns and consider period j such that $\rho_j \geq \frac{1-2k}{1+(n-j)k}$ and $\rho_{j+1}(1) < \frac{1-2k}{1+(n-j-1)k}$. In words, period j is the critical period (given that no action 0 is observed in the previous periods). If action 0 is observed in period j, the agent is hired for the remaining periods. If action 1 is observed, then the agent is never hired afterward. Higher n implies higher $n-j$; hence, there are more periods left to be played. Thus, increasing n implies the principal will hire for lower reputation levels in the critical period. This, in turn, implies that, for a given initial reputation level, the agent will be hired for an equal or more number of periods. Hence, $t(n)$ is weakly increasing in the number of periods.

The effect of change in the number of periods on the expected payoffs is ambiguous. The following has a positive effect on the continuation payoffs. As the number of interaction periods increases, t (the agent is hired for more periods) may increase. Moreover, $n-t$ may also increase or do not change. In any case, the average number of periods that the agent is hired increases. Also, if t increases, the probability that action 0 is observed also increases. The negative effect of the increase in n is that as the weights of the periods are normalized to add up to 1, each period's weight $1/n$ decreases. Even though the agent will be hired for more periods on average, the weight of each period will decrease. Hence, I can not conclude on the effect of period size on payoffs.

4.2 What happens when k increases?

The ex ante equilibrium payoffs of the players are as follows:

$$V_1^P = \frac{1}{n}\left(\rho_1\left(nk - \frac{n}{2^t}k + \frac{1}{2}t - \frac{1}{2}tk\right) + tk - \frac{1}{2}t\right)$$

$$V_1^g = \left(1 - \frac{1}{2^t} + \frac{t}{2^t n}\right)k$$

The expected payoff of both the principal and the good agent increases. It is intuitive as k denotes the value of the project. The threshold for hiring decreases. A similar reasoning as subsection 4.1 applies. In the critical period, higher k implies a higher expected continuation payoff to the principal. As the future expected payoff of the principal is higher in the critical period, he hires the agent for lower reputation levels, i.e., $\frac{1-2k}{1+(n-i)k}$ is decreasing in k. Thus, fixing an initial reputation level, increasing k parameter implies the agent will be hired for equal or more periods. Hence, t is weakly increasing in k.

If I treat k parameter as the project's size, I reach a natural conclusion as follows. The more significant (or, the more important) the project is, the higher is the hiring frequency.

4.3 When there will be reputation concerns?

If n or k is large, then there are reputation concerns. Let us start with the complementary proposition to Proposition 2 for $\rho_1 < R(n,n)$.

Proposition 3. *1. If there exists $t \in \{1,2,...,n-1\}$ such that $R(t,n) \leq \rho_1 < R(t+1,n)$ and $n > \frac{1}{k}+t$, then the agent has reputational incentive for some period in any perfect Bayesian equilibrium, i.e., $\mu_i = \nu_i = 0$ can not hold $\forall i \in N$, .*

2. If $\rho_1 < R(1,n)$ and $n \geq \frac{1}{k}+1$, then the agent has reputational incentive for some period in any perfect Bayesian equilibrium, i.e., $\mu_i = \nu_i = 0$ can not hold $\forall i \in N$, .

Proof. Appendix. □

The intuition behind Proposition 3 is straightforward. Following the analysis in Proposition 2, given $R(t,n) \leq \rho_1 < R(t+1,n)$ and $n \geq \frac{1}{k}+t$, suppose for contradiction that the agent has no reputation incentive in the initial t periods. She will not be hired from period $t+1$ onward if no action 0 is observed. Having no reputation concern means the bad agent plays action 1 in any period. Hence, the bad agent gets a payoff of $\frac{t}{n}k$. If the bad agent deviates to action 0 in any period, she gets a payoff of $\frac{1}{n}(k-1+(n-1)k)$. $n > \frac{1}{k}+t$ implies that it is a profitable deviation. Hence, there will be reputation concern for some period given $n > \frac{1}{k}+t$.

The extent of the reputation concerns depend on the relation between n,k and t. For illustration, Proposition 4 demonstrates equilibrium behavior for a specific case.

Proposition 4. *Let $\rho_1 < R(2,n)$ and $\frac{1}{k}+2 \geq n > \frac{1}{k}+1$, the following constitute the agent optimal Perfect Bayesian equilibrium:*

$$\mu_1 = 1,\ \nu_1 = \frac{1}{2^{n-3}} \frac{\rho_1 k}{(1-\rho_1)(1-2k)},\ \mu_i = 0 = \nu_i\, \forall i \in N/1$$

$$\lambda_1 = \begin{cases} 1, & \rho_1 \geq \dfrac{(1-2k)2^{n-3}}{(n-1)k(2^{n-2}-1)} \\ 0, & otw \end{cases}$$

$\lambda_2(0) = 1,\ \lambda_2(1) = 0.\ \lambda_i(0) = 1 = \lambda_i(1)\, \forall i \in N/\{2,n\}$ *and* $\lambda_n(0) = 1$.

$$\lambda_n(1) = \begin{cases} \dfrac{1}{k} - (n-2), & a_i = 1\, \forall i \in \{2,3,...,n-1\}\, in\, h_n, \\ 1, & otw \end{cases}$$

Proof. Appendix. □

The case analyzed in the Proposition 4 is a special case where the initial reputation is small and n is so that there are reputation concerns only in the first period. If the initial reputation is small, $\rho_1 < R(2,n)$, and both types of agent play the stage-game maximizing strategies, then the principal only hires the agent in the second period if action 0 is observed. $n > \frac{1}{k} + 1$ implies the bad agent will deviate to play action 0. Hence, no reputation concern in the first period can not hold in any equilibrium. Moreover, $\frac{1}{k} + 2 \geq n$ implies there are no more reputation concerns from period 2 onward. I have a discrete number of periods. I need to introduce a mixing strategy for the principal so that the agent can be indifferent between both actions. Otherwise, reputation concerns imply $\mu_i = 1 - \nu_i$. There will be no update on the reputation level. However, $\lambda_2 = 0$ if $\rho_2 = \rho_1$ as the initial reputation is sufficiently small. Hence, strategies should be so that there is an update in the reputation level, which necessitates indifference for at least one type of agent.

Note that the threshold initial reputation for hiring, $\frac{(1-2k)2^{n-3}}{(n-1)k(2^{n-2}-1)}$ is strictly less than $\frac{1}{2} - k$. It implies, if the agent has reputation concerns, then she is hired for lower initial reputation levels (for some $\frac{(1-2k)2^{n-3}}{(n-1)k(2^{n-2}-1)} \leq \rho_1 < \frac{1}{2} - k$) compared to the case with the absence of reputation concerns. Unfortunately, I can not define broader equilibrium behavior as it varies with n, k and ρ_1,

5 Endogenous Stakes

I have the same model as in Section 3. Additionally, in each period i, a player sets the relative stake of the period i decision. More precisely, in period i, $\delta_i \in [0,1]$ is the weight of all the future periods $i+1, i+2, ..., n$ while $1-\delta_i$ is the weight of period i. In other words, in each period i, the player sets the proportion of remaining decisions to be made in that period as $(1-\delta_i)$ and the proportion to leave to subsequent periods as δ_i. Thus, if I define a weight for each period, γ_i, then $\gamma_i = (1-\delta_i)\prod_{j=1}^{i-1}\delta_j$.

I assume $\delta_i \in \left[\underline{\delta}, \bar{\delta}\right] \forall i$ such that $\bar{\delta} = 1 - \underline{\delta}$ and focus on the case where $\underline{\delta}$ is arbitrarily close to 0.

The timing of the stage game in period i is altered as follows:

1. The agent or the principal set δ_i.
2. The principal makes the hiring decision. If $\lambda_i = 0$, then the game moves to the next period. Both parties get a payoff of 0.
3. If $\lambda_i = 1$, nature chooses the state $\theta_i \in \{0,1\}$.
4. The agent observes θ_i and takes the action $a_i \in \{0,1\}$.

The stage game payoffs and stage fame strategies are same with Section 3, given by the Tables 1 and 2.

Results of Lemmas 1, 2 and 3 continue to hold. In words, while playing action 0 weakly increases the agent's reputation, playing action 1 decreases it in the agent-optimal perfect Bayesian equilibrium. Hence, the good agent has no incentive to play action 1 when the state of the world is 0.

I continue to focus on the perfect Bayesian equilibria with Markovian property. In that regard, previous history (including δ_i) choices matter only in terms of their effect on the reputation of the agent. Thus, if the agent has the same reputation level after two different histories, then both the agent's and the principal's action is the same in both cases. The decisions depend on the past observations only through their effect on the reputation of the agent. When there are multiple equilibria, I use agent-optimality (and principal optimal whenever needed) for the equilibrium selection.

5.1 Agent sets stakes

I show that the agent allocates stakes so that (1) the principal hires the agent and (2) she gets maximum payoff given she is hired. This allocation enables the agent to maximize her continuation payoff.

Recall:

$$R(t,n) = \frac{(1-2k)2^{t-1}}{(1-2k)2^{t-1} + (n-t+2)k}$$

I defined $R(t,n)$ to set t corresponding to initial reputation level. Now, I will define general $R(i,t,n)$ for period i:

$$R(i,t,n) = \frac{(1-2k)2^{t-1}}{(1-2k)2^{t-1} + (n-t-i+3)k}$$

If $\rho_i > R(i,t,n)$ and the agent has no reputation concern in periods $i, i+1, ..., i+t-2$, then, by Bayes rule, $\rho_i + t - 1 \geq 1-2k$. Following the same reasoning as in Section 4, the agent's reputation stays above $1-2k$ for at least t periods (starting from period i). Hence, if $\rho_i \geq R(i, n-i+1, n)$ and the agent does not take the costly reputation-action, then $\rho_n \geq 1-2k$. In that case the agent's choice of δ is arbitrary and the agent plays stage-game maximizing strategy in each period $i, i+1, ..., n$ in the agent-optimal equilibrium. She gets the maximum possible payoff.

On the complementary case, $\rho_i < R(i, n-i+1, n)$ implies that $\lambda_j(1) = 0$ for some $j > i$ in the agent-optimal equilibrium. The future expected payoff of the agent in period i is strictly less than k. Therefore, the agent sets δ_i as low as possible, i.e. $\underline{\delta}$.

The agent's choice of δ_i depends on ρ_i as follows. For $\rho_i \geq 1-2k$, the principal hires the agent irrespective of the allocation of stakes, because the principal obtains a non-negative payoff in period i (if there is no reputation concern in period i, $U_i^P = k - \frac{1}{2} + \frac{1}{2}\rho_i$). Consider the extreme case where $\underline{\delta} = 0$ and the agent sets $\delta_i = \underline{\delta}$. The interaction becomes one-shot as there is no stakes left to the future. From the analysis before, the principal hires the agent if $\rho_i \geq 1-2k$. In that case, it is optimal for the agent to set δ as low as possible, i.e. leave the minimum possible stakes to the future.

On the other hand, if the reputation is below $1-2k$, then the minimum $\delta_i = \underline{\delta}$ leads to negative continuation payoff to the principal (by definition, $\underline{\delta}$ is sufficiently small). Hence, the agent sets minimum δ_i value which leads to a non-negative continuation payoff for the principal, which corresponds to $\delta_i^{'} = \dfrac{1-2k-\rho_i}{1-2k-\rho_i+\rho_i k}$. Again, the agent chooses the minimum possible δ value. For

$\rho_i \geq \dfrac{1}{2} - k$, $\delta_i^{'} \leq \dfrac{1}{1+k}$. Thus, $\mu_i = 0$ and $\nu_i = 0$ is an equilibrium strategy.

Consider $\rho_i < \dfrac{1}{2} - k$. The lowest δ_i leading to a non-negative payoff for the principal is greater than $\dfrac{1}{1+k}$. Therefore, no reputation concern is not an equilibrium strategy implied by the sequential rationality of the agent. Both type of agent would deviate to play action 0. This implies that, given $\rho_i < \dfrac{1}{2} - k$, the agent may only be hired in period i if the agent has reputation concerns. Suppose the principal hires the agent and the agent has reputation concern in period i. Consider the equilibrium path following period i. The agent's payoff in period $i+1$ is decreasing in $\underline{\delta}$. Define $\tilde{\delta}_i$ as agent's choice of δ_i which implements reputation concerns in the equilibrium. As V_{i+1}^{g} is decreasing in $\underline{\delta}$; lower $\underline{\delta}$ implies lower $\tilde{\delta}_i$. To see the intuition, consider the following. If the lower limit of the stakes the agent can postpone to future (δ_{i+1}) increases, the agent's payoff decreases in period $i+1$. From the period i agent's point, the future becomes less appealing. Hence, to have reputation concerns, δ_i should be higher ($\tilde{\delta}_i$ increases). Next thing to realize is that the principal gets a negative payoff in period i as there are reputation incentives. Hence, for the principal to hire the agent, equilibrium δ_i needs to be above a certain level. In words, the higher δ_i, the lower stakes are attached to period i in which the principal obtains a negative payoff. Returning to the sequential rationality constraint of the agent; for indifference δ_i to be high, $\underline{\delta}$ should be above a certain level. Overall, for the principal to hire agent in period i where $\rho_i < \dfrac{1}{2} - k$ (and the agent has reputation incentives) , equilibrium δ_i should be above a certain level (which implies $\underline{\delta}$ should be above a certain level). However, as I focus on sufficiently small $\underline{\delta}$, I conclude that the principal does not hire the agent in such case.

Remark 1. *If $n < \frac{1}{k} + 1$, the agent has no reputation concern in the equal-stakes model. In such case, $\frac{1}{2} - k < R(1,n)$. It implies the agent is hired for lower reputation level if she sets the stakes. On the other hand, consider the special case in Proposition 4. The agent is hired if $\rho_1 \geq \dfrac{(1-2k)2^{n-3}}{(n-1)k\left(2^{n-2}-1\right)}$. A simple calculation shows that $\dfrac{(1-2k)2^{n-3}}{(n-1)k\left(2^{n-2}-1\right)} < \frac{1}{2} - k$. It implies that the agent is hired for lower reputation levels if the stakes are equally allocated. Hence, the agent's ability to set stakes hurts her equilibrium payoff.*

In any period i, the agent is hired if and only if $\rho_i \geq \frac{1}{2} - k$. Define

$$P(1,t,n) = \begin{cases} \frac{(1-2k)2^{t-1}}{(1-2k)2^{t-1}+1+2k}, & t \in \{1,...,n-2\} \\ R(1,t,n), & t = n-1 \end{cases}$$

If $P(1,t,n) \leq \rho_1 < P(1,t+1,n)$, $\mu_i = \nu_i = 0$ and $a_i = 1$ for $i \in 1,2,...,t-1$, then $\rho_t \geq \frac{1}{2} - k$ and $\rho_{t+1}(1) < \frac{1}{2} - k$. In words, if the agent does not hold reputation concerns, then she is hired for exactly t periods if no action 0 is observed. The principal hires the agent in period $i < n$ if $\rho_i \geq \frac{1}{2} - k$. Hence, given $P(1,t,n) \leq \rho_1 < P(1,t+1,n)$, the agent is hired for t periods if no action 0 is observed. The range of reputation is defined so that, $\rho_t \geq \frac{1}{2} - k$ and $\rho_{t+1} < \frac{1}{2} - k$ if no action 0 is observed for $t < n-1$. This implies, the agent is hired in period t but not in period $t+1$ onward. Note that $P(1,1,n)$ corresponds to $\frac{1}{2} - k$, hence, the agent is not hired if $\rho_1 < P(1,1,n)$.

However, in period n, the principal hires the agent if the reputation is not below $1-2k$, because there are no more periods left to be played. If $P(1,n-2,n) \leq \rho_1 < P(1,n-1,n)$, the agent is hired for $n-1$ periods if no action 0 is observed. Hence, $\rho_{n-1} \geq \frac{1}{2} - k$ and $\rho_n < 1-2k$. Thus, the agent is hired in period $n-1$, but not hired in period n. As there is no reputation concern, $\rho_{i+1}(0) = 1$ and $\rho_{i+1}(1) = \frac{\rho_i}{2-\rho_i}$ for all i.

If $\rho_1 \geq P(1,n,n)$, the agent has no reputation concerns and is hired in all periods irrespective of the observed action. By Bayes' rule, $\rho_n \geq 1-2k$ even if no action 0 is observed. Both types of agent get the maximum possible payoff - k - from the project. Hence, this equilibrium is not payoff-dominated. The agent sets δ_i arbitrarily as she is hired in every period.

Consider the following proposition.

Proposition 5. *Agent-optimal PBE is as following.* $\mu_i = 0 = \nu_i$ *and*

$$\delta_i \in \begin{cases} [\underline{\delta}, \bar{\delta}], & \rho_i \geq R(i,n-i+1,n) \\ \underline{\delta}, & 1-2k \leq \rho_i < R(i,n-i+1,n) \\ \delta_i^{'} = \frac{1-2k-\rho_i}{1-2k-\rho_i+\rho_i k}, & \frac{1}{2} - k \leq \rho_i < 1-2k \end{cases}$$

$$\lambda_i = \begin{cases} 0, & \rho_i < \frac{1}{2} - k \\ 1, & \rho_i \geq \frac{1}{2} - k \end{cases}$$

Proof. Appendix. □

The stakes start large and gradually decrease. For $P(1,t,n) \leq \rho_1 < P(1,t+1,n)$, the agent is hired for at least t periods. After s periods, the reputation of agent falls below $1-2k$ if no action 0 is observed. Note that $s < t$ as the agent is also hired when $\rho_i \leq 1-2k$. The evolution of stakes are as follows:

$$\Gamma = \{(1-\underline{\delta}), (1-\underline{\delta})\underline{\delta}, (1-\underline{\delta})\underline{\delta}^2, ..., (1-\underline{\delta})\underline{\delta}^{t-3}, \underline{\delta}^{t-2}\left(1-\delta'_{t-1}\right), \underline{\delta}^{t-2}\delta'_{t-1}\left(1-\delta'_t\right)\}$$

The agent may survive for more than 2 periods after the reputation falls below $1-2k$. It solely depends on the k parameter. In the distribution above, it was implicitly assumed that $k > 1/6$, so that if $\rho_{t-1} < 1-2k$ and action 1 is observed for two consecutive periods (periods $t-1,t$), then $\rho_{t+1} < 1/2 - k$. For any k parameter, starting large and gradual decrease result holds.

The ex-ante equilibrium payoffs of the players are as follows:

$$V_1^P = \rho_1 \underbrace{\left(1-\frac{1}{2^t}\right)}_{prob[a_i=0\, for\, some\, i]} k + \rho_1 \underbrace{\left(\frac{1}{2^t}\right)}_{prob[a_i=1\, for\, all\, i]} \left(\underbrace{(1-\underline{\delta}) + (1-\underline{\delta})\underline{\delta} + ...}_{\text{s periods where reputation is above 1-2k}} + \underbrace{\underline{\delta}^{t-2}\left(1-\delta'_{t-1}\right) + \underline{\delta}^{t-2}\left(1-\delta'_t\right)\delta'_{t-1}}_{\text{2 or more periods where reputation is below 1-2k}} \right) k$$

$$+ (1-\rho_1)\left((1-\underline{\delta}) + (1-\underline{\delta})\underline{\delta} + ... + \underline{\delta}^{t-2}\left(1-\delta'_{t-1}\right) + \underline{\delta}^{t-2}\left(1-\delta'_t\right)\delta'_{t-1}\right)\left(k-\frac{1}{2}\right)$$

$$V_1^g = \left(1-\frac{1}{2^t}\right)k + \frac{1}{2^t}\left((1-\underline{\delta}) + (1-\underline{\delta})\underline{\delta} + ... + \underline{\delta}^{t-2}\left(1-\delta'_{t-1}\right) + \underline{\delta}^{t-2}\left(1-\delta'_t\right)\delta'_{t-1}\right)k$$

5.2 Principal sets stakes

Following the previous analyses, if $\rho_i \geq R(i, n-i+1, n)$, then the agent has no reputation concern in periods $i, i+1, ..., n$, in the agent-optimal equilibrium. If the agent plays stage-game payoff maximizing action in each period, then by Bayes' rule, $\rho_j \geq 1-2k \, \forall j \in \{i, i+1, ..., n\}$. Hence, the agent is hired every period and gets the payoff of k, which is the agent-optimal equilibrium. Therefore, the principal's choice of δ does not affect the agent's strategy. The principal hires the agent irrespective of the observed action; hence, the principal arbitrarily sets δ_i. The following arguments focus on the lower reputation levels.

I showed that the agent prefers to start the interaction large. This section analyzes the framework where the principal is setting the relative stakes. The principal prefers to start small. If δ_i is such that no type has a reputational incentive, then the agent's expected continuation payoff is decreasing in δ_i. The reasoning is that the agent gets the payoff of k in period i, whereas the future expected continuation payoff is weakly less than k. Consider the principal. For such δ_i, as there will be the complete revelation of the type of the good agent after action 0, the future expected continuation payoff of the principal is higher than the payoff in period i. Thus, the principal prefers to increase δ_i until the point where the reputational incentives emerge.

Note that $\delta_i < \frac{1}{1+k}$ is the sufficient condition for the absence of reputation concerns. In other words, if period i receives more than $\frac{k}{1+k}$ proportion of the remaining decisions, then the agent has no reputation concern in any equilibrium. It follows from the sequential rationality of the agent. In any period i, the maximum payoff agent can get from taking the costly reputation-driven action is given by $(1-\delta_i)(k-1) + \delta_i(k)$. On the other hand, she can get a payoff weakly greater than $(1-\delta_i)k$ if she plays action 1. Hence, if $\delta_i < \frac{1}{1+k}$, then the agent plays stage-game maximizing action. Thus, the principal deviates to set $\delta_i = \frac{1}{1+k}$. More precisely, I show that the principal sets $\delta_i \geq \frac{1}{1+k}$ in every period. In words, principal leaves higher stakes to the future periods, i.e. starts small.

Proposition 6. *There is no equilibrium where $\rho_i < R(i, n-i+1, n)$ and $\lambda_i = 1$ and $\delta_i < \frac{1}{1+k}$.*

Proof. Appendix. □

Remark 2. *Consider period 1 with endogenous stakes, $\gamma_1 = 1 - \delta_1$. $\delta_1 < \frac{1}{1+k}$ implies that $\gamma_1 > \frac{k}{1+k}$. Recall the sufficient condition for the absence of reputation concerns in the benchmark model stated in Proposition 1. I showed that if $n < \frac{1}{k} + 1$, then the agent has no reputation concern in any equilibrium. Note that, in benchmark model, each period receives a stake of $\frac{1}{n}$, i.e. $\gamma_i = \frac{1}{n} \forall i$. The condition $\delta_1 < \frac{1}{1+k}$ corresponds to $n < \frac{1}{k} + 1$ for $\gamma_1 = \frac{1}{n}$.*

Next, I define the agent-optimal equilibrium when there is a sufficiently small probability of the agent being a good type. If the initial reputation is low, then the principal starts the interaction with low stakes. I define sufficiently small initial reputation as the lowest initial reputation level for which the agent is hired in the equilibrium. By doing so, the principal gradually updates his belief about the agent's type. In the equilibrium, the whole interaction (except for period n) serves as the *evaluation phase*.

I define *evaluation phase* as follows: in any period of the evaluation phase, the principal continues to hire the agent as long as action 0 is observed, if otherwise, action 1 is observed in any period, the principal ends the relationship. $\nu_i > 0$ must hold in the evaluation phase except for the last period. If $\nu_i = 0$ for some i, then $\rho_{i+1}(0) = 1$ which means the agent is hired for the remaining periods if $a_i = 0$ is observed. It contradicts the definition of the evaluation phase. $\nu_i = 0$ can hold only in the last period of the evaluation phase.

Consider the following cases to understand how the principal sets the length of the evaluation phase strategically. Case 1: the evaluation phase involves the first s periods, i.e., given no action 1 is observed in the previous periods, the agent is hired for periods $s+1, s+2, ..., n$ irrespective of the action observed. Case 2: evaluation phase involves the first t periods such that $s < t$. In case 1, there are more periods left to be played after the evaluation phase. Thus, if each period received the same stake, the agent would have more reputation concerns in case 1. Hence, the principal needs to put higher stakes per the period of the evaluation phase in case 1 compared to case 2. Thus, the smaller the evaluation phase, the higher stakes the principal risks in each period. The intuition is that the shorter the evaluation phase, the higher stake each period should receive to incentivize the agent. But it means risking high stakes, which leads to a lower expected continuation payoff. If the initial reputation is sufficiently small, the evaluation phase involves all periods except the last one.

Building reputation is costlier for the bad agent except for the period $n-1$. In period $n-1$, both types' future expected payoff is the same; hence, they have the same sequential rationality constraint. For earlier periods, playing action 0 costs the same to both types, whereas the good agent has a higher future expected payoff. By taking the costly action, the good agent gets more chances to improve her reputation costlessly in the future (when the true state is 0 and she plays accordingly). So the good agent values future interaction more. It implies investing in her reputation is less costly for her. Thus, the sequential rationality constraints of both types differ when there is more than one period left to be played. In other words, if the stakes are allocated so that $\nu_i > 0$, then $\mu_i = 1 \; \forall i \in N/\{n-1, n\}$.

The principal sets δ_i values so that the bad agent is indifferent between two actions in each period and $\nu_i \in (0,1) \forall i \in N/\{n-1, n\}$. If $a_i = 1$ is observed in any period, the principal is sure that the agent is the bad type. Whereas the reputation after $a_i = 0$ evolves at just enough pace so that the agent is hired for one more period.

Remark 3. *In the equilibrium, there exists* $\underline{\nu}_i$ *and* $\bar{\nu}_i$ *for all* $i \in N$ *such that* $\nu_i \in [\underline{\nu}_i, \bar{\nu}_i]$ *is consistent with equilibrium beliefs. For* $\nu_i \in [\underline{\nu}_i, \bar{\nu}_i]$, $\rho''_{i+1} \geq \rho_{i+1}(0) > \rho'_{i+1}$, *in words, reputation stays in the lowest range. As the principal's payoff decreases in* ν_i *for all i, the bad agent plays action 0 with the lowest possible probability that satisfies equilibrium conditions in the principal-optimal equilibrium. Thus,* $\nu_i = \underline{\nu}_i$ *and the reputation evolves as* $\rho_{i+1}(0) = \rho''_{i+1} \; \forall i \in N$. *The reputation evolves in just enough pace so that* $\rho_{n-1}(0) = \dfrac{1-2k}{1-k}$.

Proposition 7 formalizes the equilibrium behavior.

Proposition 7. *There exists* $\rho'_1 < \rho''_1$ *such that for* $\rho'_1 \leq \rho_1 < \rho''_1$, *the following strategies constitute the principal-optimal perfect Bayesian equilibrium:*

$$\nu_i \in (0,1),\; \mu_i = 1 \textit{ for } i \in N/\{n, n-1\},\; \nu_{n-1} = 0,\; \mu_{n-1} \in [0,1],\; \nu_n = 0,\; \mu_n = 0$$

$$\delta^*_i = \frac{\sum_{h=0}^{n-i-1} k^h}{\sum_{h=0}^{n-i} k^h},\; \lambda_i(0) = 1 \textit{ and } \lambda_i(1) = 0 \textit{ for all } i \in N/1.\; \lambda_1 = 1$$

Proof. Appendix. □

In period $n-1$, $\delta_i = \dfrac{\sum_{h=0}^{n-i-1} k^h}{\sum_{h=0}^{n-i} k^h} = \dfrac{1}{1+k}$ and the good agent is also indifferent between two actions and she mixes. The sequential rationality constraints of both types are the same in $n-1$. There is one-period play left, so the future expected payoff of both types is k if the game is not terminated and 0 if it is terminated. As the principal's continuation payoff is decreasing in v_i, then $v_{n-1} = 0$ in the principal-optimal equilibrium. I see that if I start with small enough initial reputation, principal starts with low γ and increases it gradually through interaction, which is in accordance with the "starting small" literature - $\gamma_i = \frac{k^{n-i}}{\sum_{h=0}^{n-1} k^h}$.

6 Conclusion

I provided an analysis of repeated interaction between a principal and an agent involving different allocation of relative stakes. I borrowed the model from Ely and Välimäki (2003), started with providing a finite version of the model offered by EV, and showed that as the number of periods converges to infinity, equilibrium behavior converges to EV.

The main contribution of this paper is introducing endogenous stakes. I showed that the player's optimal stake allocation depends on their position on the asymmetric information problem. The agent is informed, and there is no uncertainty about the principal's preferences. Hence, the agent prefers to start the interaction large so that there are no reputation concerns. Starting large may or may not be to the advantage of the agent. If the agent does not have reputation concerns in the equal-stakes allocation, then starting large has a two-fold advantage for the agent. First, the agent gets a higher payoff in starting-large allocation. Second, the agent is hired for lower initial reputation levels for which she is not hired if the stakes are equally allocated.

Nevertheless, the agent's ability to endogenously set the stakes may affect negatively. I show the existence of cases in which the agent is hired if the stakes are equally allocated but not hired if the agent sets the stakes. The intuition is as follows. If the agent starts large, it is risky for the principal to hire the agent if the initial reputation is sufficiently low. On the other hand, suppose the agent starts with small stakes. Then there will be reputation concerns. The agent can not commit to an allocation of stakes at the beginning of the interaction. Screening occurs in the initial period(s), and the principal continues to hire the agent if the reputation goes above a certain level. However, after the reputation level increases, the agent puts the highest possible stakes in every consecutive period. The principal obtains a zero or sufficiently small continuation payoff in any such period. In addition, the principal gets a negative period payoff in any period involving reputation concerns. Thus, in total, the principal expects a negative continuation payoff in the first period if the agent has a sufficiently small initial reputation level. In such cases, exogenously set stakes may be preferred to endogenous stakes.

The principal does not know the agent's preferences and is uninformed about the state of the world. For him, starting large is equivalent to starting a risky business with high stakes. On the other hand, starting small is appealing to the principal for the following reasons:

1. As larger stakes are left to the future periods, the agent will hold reputation incentives. The principal benefits from the reputation incentives of the agent as it helps him update his beliefs on the agent's type.
2. As a smaller stake is allocated to the current period, the principal risks a small proportion of the project. If the agent is a bad type, then her harm to the project will be limited.

Hence, the principal prefers to start small and always benefits from the ability to set the stakes endogenously.

Appendix

Proof. Lemma 1

Let $\mu_i(0) \equiv \mu_i(h,1,0)$ and $\mu_i(1) \equiv \mu_i(h,1,1)$ denote the probability with which the good agent plays action 0 when the state of world is 0 and 1, respectively.

I first show that there are no reputation concerns in the last three periods, i.e., both type of agent play the stage-game maximizing actions (the action matching the state of world for the good agent, and action 1 for the bad agent). If an agent plays differently, the maximum payoff she can get in the last three periods is $k-1+2k$. On the other hand, stage-game maximizing actions lead to the payoff weakly greater than k. $k<\frac{1}{2}$ implies $k>k-1+2k$. Thus, in any equilibrium, $\mu_i(0)=1$, $\mu_i(1)=0$, $\nu_i=0$ for $i\in\{n-2,n-1,n\}$. It in turn implies by Bayes' rule, $\rho_{i+1}(0)\geq\rho_{i+1}(1)$ for $i\in\{n-2,n-1\}$.

Consider period $n-3$. As there are no reputation concerns in the last three periods, agent prefers higher ρ_{n-2}. Suppose $\rho_{n-2}(1)>\rho_{n-2}(0)$. Then $\nu_{n-3}=0$ as the agent prefers higher reputation level in period $n-2$. If good agent plays action 0 with a positive probability, then $\rho_{n-2}(0)=1$ contradicting $\rho_{n-2}(1)>\rho_{n-2}(0)$. Thus, $\rho_{n-2}(1)>\rho_{n-2}(0)$ can only hold if $\mu_{n-3}(0)=0$, $\mu_{n-3}(1)=0$, i.e. both agents play action 1 with probability 1. $\rho_{n-2}(1)=\rho_{n-3}$ and $\rho_{n-2}(0)$ is out of equilibrium path, hence I can set $\rho_{n-2}(1)>\rho_{n-2}(0)$.

Sequential rationality of the agent implies that it can only be an equilibrium strategy if given $a_{n-3} = 1$, the agent is hired for at least two periods. Suppose not. The second best alternative: the agent is hired in period $n-2$, but not hired in period $n-1$ if $a_{n-2} = 1$ is observed. Consider $\theta_{n-3} = 0$. Good agent gets payoff of $k - 1 + k + \frac{1}{2}k$ if she plays $a_{n-3} = 1$. On the other hand, playing action 0 yields payoff k, which is strictly greater. Hence, the agent will deviate. Now consider ρ_{n-3} such that, if the reputation is not updated in period $n-2$ (i.e. $\rho_{n-2}(1) = \rho_{n-3}$), then the agent is hired also in period $n-1$ irrespective of the action observed. Now, given $\theta_{n-3} = 0$, playing action 0 yields a payoff weakly greater than $k - 1 + 2k + \frac{3}{4}k$ which may be greater than k for some $k < \frac{1}{2}$. Hence, $\nu_{n-3} = 0$, $\mu_{n-3}(0) = 0$, $\mu_{n-3}(1) = 0$ can be an equilibrium.

Define $m \geq 2$ as the number of periods for which the agent is hired following $a_{n-3} = 1$ if no action 0 is observed in the further periods. Then, the good agent's continuation payoff:

$$V^g_{n-3} = \left(k - \frac{1}{2}\right) + mk + \left(1 - \frac{1}{2^m}\right)(3 - m)k \tag{1}$$

Now consider the following strategies. $\mu_i(0) = 1$, $\mu_i(1) = 0$, $\nu_i = 0$ for $i \in \{n-3, n-2, n-1\}$. The agent is hired for at least m periods if no action 0 is observed. The reasoning is as following. If the agent is hired in period i when the reputation is ρ', then the agent is also hired in period $i-1$ with reputation level ρ'. As the number of interaction periods increase, the principal hires the agent for weakly lower reputation levels. The good agent gets the payoff of:

$$V^g_{n-3} \geq mk + \left(1 - \frac{1}{2^m}\right)(4 - m)k \tag{2}$$

This strategies constitute an equilibrium. The only profitable deviation may be to deviate to play action 0 in period $n-3$. Deviation yields the payoff of $k - 1 + 3k$, which is strictly less than 2, given $m \geq 2$ and $k < \frac{1}{2}$. Bad agent also has no profitable deviation. She gets payoff weakly greater than mk, which is strictly greater than $k - 1 + 3k$.

Note that 2 is strictly higher than 1. As there is an equilibrium yielding a higher payoff to the agent, both type playing action 1 can not be the agent-optimal equilibrium. Hence, in the agent-optimal equilibrium $\rho_{i+1}(0) \geq \rho_{i+1}(1)$ for $i \in \{n-3, n-2, n-1\}$.

Next, I will show that the agent's payoff is weakly increasing in reputation in period $n-3$. Define ρ'_{n-2} such that $\lambda_{n-2} = 1$ if $\rho_{n-2} \geq \rho'_{n-2}$. If ρ_{n-3} is so that $\dfrac{\rho_{n-3}}{2 - \rho_{n-3}} \geq \rho'_{n-2}$, then the agent-optimal equilibrium is where there are no reputation concerns. In this equilibrium, good agent gets the payoff weakly greater than $2k+\frac{3}{4}2k$. The higher ρ_{n-3}, the agent is hired for weakly more periods, hence gets a weakly higher payoff. This is the highest payoff agent can get in any equilibrium. In any equilibrium with reputation concerns, the good agent can get a payoff weakly less than $k-1+3k$, which is strictly less.

Now consider $\rho_{n-3} < \dfrac{2\rho'_{n-2}}{1+\rho'_{n-2}}$. No reputation can not be equilibrium strategy in this case because $\rho_{n-2}(0) = 1$ and the bad agent will deviate to play action 0. Thus, there will be reputation concerns in this case and the maximum payoff the agent can get is given by $k-1+3k$. Observe that it is strictly less than the payoff of the agent when $\rho_{n-3} \geq \dfrac{2\rho'_{n-2}}{1+\rho'_{n-2}}$. Also, the payoff is weakly increasing in reputation for $\rho_{n-3} \geq \dfrac{2\rho'_{n-2}}{1+\rho'_{n-2}}$. Thus, the agent's payoff is weakly increasing in reputation in period $n-3$. Hence, agent's payoff in period $n-2$ increasing in ρ_{n-2} implies $\rho_{n-2}(0) \geq \rho_{n-2}(1)$ and the agent's payoff in period $n-3$ is increasing in ρ_{n-3}.

I finalize the proof with induction method. Consider period i such that $\rho_{j+1}(0) \geq \rho_{j+1}(1)$ and $\mu_j(0) = 1$ for all $j > i$. Suppose $\rho_{i+1}(1) > \rho_{i+1}(0)$. Following the same reasoning above, it can only hold if both agent play action 1.

Case 1

No reputation concern in any following period is an equilibrium. More formally, consider ρ_i is such that the agent is hired for t periods if no action 0 is observed. If $tk \geq k-1+(n-i)k$, then no agent has a profitable deviation, hence it is an equilibrium.

$\rho_{i+1}(1) > \rho_{i+1}(0)$ can only hold if both agents play action 1 with probability 1. Following the same steps above, for this to be an equilibrium formation, ρ_i must be above certain level so that the agent is hired for $m \geq 1$ periods. Hence,

$$V_i^g = \left(k - \frac{1}{2}\right) + mk + \left(1 - \frac{1}{2^m}\right)(n-i-m)k \tag{3}$$

on the other hand, in the equilibrium where there are no reputation concerns, the agent get:

$$V_i^g \geq mk + \left(1 - \frac{1}{2^m}\right)(n-i-m)k \tag{4}$$

4 is strictly higher than 3, hence the no reputation yields a higher payoff to the agent.

Case 2

No reputation concern in any following period is not an equilibrium. More formally, consider ρ_i is such that the agent is hired for t periods if no action 0 is observed. If $tk > k-1+(n-i)k$, then the agent will deviate to play action 0 in period i.

Again $\rho_{i+1}(1) > \rho_{i+1}(0)$ can only hold if both agents play action 1 with probability 1. Following the same steps above, for this to be an equilibrium formation, ρ_i must be above certain level so that the agent is hired for $m \geq 1$ periods. The difference is that there will be reputation concerns in the period $i+1$ (as $\mu_{i+1}(0) = 1$, $\nu_{i+1} = 0$ can not be an equilibrium strategy because it implies $\rho_{i+2}(0) = 1$) . The maximum payoff the good agent can get in this formation:

$$V_i^g = \left(k - \frac{1}{2}\right) + \left(k - \frac{1}{2}\right) + (n-i-m)k \tag{5}$$

on the other hand, consider the following equilibrium. The agent may screen her type in period i rather than in period $i+1$:

$$V_i^g \geq \left(k - \frac{1}{2}\right) + (n-i-m)k \tag{6}$$

6 is strictly higher than 5, hence screening earlier yields a higher payoff to the agent.

□

Proof. Lemma 2

Suppose for contradiction that there is an equilibrium where $\mu_i(0) < 1$ for some i. From Lemma 1, $\rho_{i+1}(0) > \rho_{i+1}(1)$ and $V_{i+1}^g(.|a_i = 0) \geq V_{i+1}^g(.|a_i = 1)$. Hence, the agent will deviate to $a_i = 0$ and increase both U_i^g and V_{i+1}^g. □

Proof. Lemma 3

From Lemma 1, $\rho_{i+1}(0) > \rho_{i+1}(1)$ implies $\nu_i < \dfrac{1+\mu_i(1)}{2}$. □

Proof. Proposition 1

Suppose for contradiction that there is an equilibrium where $\mu_i > 0$ for some i. The maximum continuation payoff the agent can get from playing action 0 is

$$k-1+(n-i)k$$

$k<\frac{1}{n-1}$ implies the agent will deviate to play action 1 and get the payoff of k. □

Proof. Lemma 4

In any period, the principal hires the agent if the expected continuation payoff is non-negative. Consider period j so that ρ_j is in the lowest range for which the agent is hired. It implies that if action 1 is observed, then $\lambda_{j+1}(1)=0$. Thus, the expected continuation payoff of the principal in period j is given by:

$$V_j^P=\left(k-\frac{1}{2}+\frac{1}{2}\rho_j\right)+\frac{1}{2}\rho_j(n-j)k$$

where $\frac{1}{2}\rho_j$ denotes the probability of observing $a_j=0$. This payoff is non-negative for $\rho_j\geq\frac{1-2k}{1+(n-j)k}$, hence $\frac{1-2k}{1+(n-j)k}$ is the lowest reputation level for which the agent is hired. □

Proof. Proposition 2

1. First, observe that $\mu_i=0\,\&\,\nu_i=0$ for all i is an equilibrium strategy for $\rho_1\geq\frac{(1-2k)2^{n-2}}{(1-2k)2^{n-2}+k}$. By Bayes' rule, $\rho_n(1)\geq 1-2k$ and the agent is hired in every period irrespective of the history. The agent has no profitable deviation as deviation to play action 0 in any period i leads to the continuation payoff of $(n-i+1)k-1$ which is less than $(n-i+1)k$.

   The next thing is to show agent-optimality. Note that $\rho_{i+1}(1)$ is decreasing in ν_i, hence irrespective of the bad agent's strategy, the reputation will be higher than $1-2k$ in all periods if $\mu_i=0$ for all i. Thus, the good agent's best response to any ν_i is $\mu_i=0$, because any path where $\mu_i>0$ leads a strictly less expected continuation payoff. Given $\mu_i=0$ for all i, the same reasoning applies to the bad agent too. In any period i, $\nu_i>0$ leads to a strictly less continuation payoff.

2. Consider period t. Following the reasoning in Proposition 1, $n<\frac{1}{k}+t$ implies the agent has no reputation concern in all the remaining periods. Thus, $\mu_i=\nu_i=0$ for $i\in t,t+1,...,n$.

From Lemma 4, the agent is hired in period t if $\rho_t \geq \frac{1-2k}{1+(n-t)k}$. By Bayes' rule, $\rho_1 \geq R(t,n)$ implies $\rho_t \geq \frac{1-2k}{1+(n-t)k}$ if $\mu_j = 0 \, \forall j < t$.

Final step is to show $\mu_j = 0 \, \forall j < t$. Consider period t and a history h such that $\rho_t \geq \frac{1-2k}{1+(n-t)k}$. Sequential rationality constraint of the good agent implies $\mu_t = 0$ irrespective of the bad agent's strategy. Agent gets a minimum payoff of k, and she can get maximum payoff of $k-1+(n-t)k$ by playing action 0 with a positive probability when $\theta_t = 1$. $n < \frac{1}{k}+t$ implies the agent has no incentive to set $\mu_t > 0$. The same reasoning applies to the bad agent, hence $\nu_t = 0$. Continuing with backward induction, I observe that good agent has no incentive to set $\mu_j > 0$ for any $j < t$. Bad agent follows.

3. Following Proposition 1, $n < \frac{1}{k}+1$ implies the agent has no reputation concern in any period. From Lemma 4, the principal's expected payoff in the first period is given by:

$$V_1^P = \left(k - \frac{1}{2} + \frac{1}{2}\rho_1\right) + \frac{1}{2}\rho_1(n-1)k$$

which is non-negative if $\rho_1 \geq R(1,n)$. Hence the agent is not hired for the lower initial reputation levels.

This completes the proof. □

Proof. Proposition 3

Directly follows Proposition 2. □

Proof. Proposition 4

I will prove in three steps.

Step 1.

Claim: $\nu_1 \in (0,1)$ must be in any informative PBE.

$\nu_1 = 0$ can not be an equilibrium strategy. Suppose for contradiction that $\nu_1 = 0$. Then $\rho_2(0) = 1$ by Bayes' rule. Hence, $V_2^b(.|a_1 = 0) = k+k+\ldots+k$. $n > \frac{1}{k}+1$ implies $k-1+k+k+\ldots+k > k$, hence the sequential rationality implies the agent will deviate to play action 0.

$v_1 = 1$ can not be an equilibrium strategy. Suppose $v_1 = 1$. Then the good agent plays action 1 with a positive probability after some θ in any informative equilibrium. Then $\rho_2(1) = 1$, bad agent deviates to play action 1.

Step 2.

Claim: $\mu_1 = 1$.

Note that the agent is not hired after action 1 and hired for some periods if action 0 is observed in the period 1. It implies $V_1^g(.|a_1 = 0) \geq V_1^b(.|a_1 = 0)$. The reasoning is that for each period that the agent is hired, there is $1/2$ probability for which the good agent can invest in her reputation costless - when the state of world is 0. Thus, the good agent has a higher reputational incentive compared to the bad agent. Suppose $V_1^g(.|a_1 = 0) = V_1^b(.|a_1 = 0)$. It only holds if the agent is hired in all periods in the future if action 0 is observed. This is the only case where both type of agent has the same sequential rationality constraint. However, $n > \frac{1}{k} + 1$ implies that bad agent will deviate to play action 0 with probability 1. But then $\rho_2(0) = \rho_1$, contradiction. Thus, $V_1^g(.|a_1 = 0) \neq V_1^b(.|a_1 = 0)$. $v_1 \in (0,1)$ implies $\mu_1 = 1$ in the equilibrium.

Step 3.

Next, I will define the principal optimality. Note that the expected continuation payoff of the principal is decreasing in v_1 as $\rho_2(0)$ is decreasing in v_1. In the principal-optimal equilibrium v_1 is equal to the lowest value satisfying the equilibrium conditions. The equilibrium with the lowest v_1, hence the highest $\rho_2(0)$ is the principal-optimal equilibrium. I will show that the equilibrium with the lowest v_1 is also agent-optimal. The highest $\rho_2(0)$ is such that the principal will hire the agent for all the remaining periods irrespective the observed action in each period. But following the reasoning above, then bad agent would deviate to play action 1 with probability 1. The second best alternative is that in the history h_n in which $a_i = 1 \ \forall i \in N/\{1,n\}$,$\rho_n(1) = 1-2k$. In words, if the action 1 is observed in each period, then the principal is indifferent between hiring the agent or not in period n. Bayes' rule implies if $\rho_2 = \frac{(1-2k)2^{n-3}}{(1-2k)2^{n-3}+k}$, then $\rho_n(a_{n-1} = 1|a_j = 1, j \in \{2,..,n-2\}) = 1-2k$. Hence, v_1 will be so that

$$\rho_2(0) = \frac{\rho_1}{\rho_1 + (1-\rho_1)v_1} = \frac{(1-2k)2^{n-3}}{(1-2k)2^{n-3}+k}$$

$\Rightarrow \nu_1 = \dfrac{\rho_1 k}{2^{n-3}(1-2k)(1-\rho_1)}$ Sequential rationality constraint of the bad agent becomes:

$$k = k - 1 + k + k + ... + k + \alpha k$$

where α stands for $\lambda_n(1)$ given action 1 is observed in each period except period 1. It implies $\alpha = \dfrac{1}{k} - (n-2)$. Note that, as $\mu_1 = 1$ in any equilibrium, the equilibrium with the highest $\rho_2(0)$ is also optimal for the good agent. With same reasoning, as $\nu_1 \in (0,1)$ in the agent-optimal equilibrium, thus $V_1^b = k$. Hence, the above defined equilibrium is also optimal for the bad agent. □

Proof. Proposition 5

1) Follows from Proposition 2. As the agent is hired for every period, she is indifferent between different allocation of stakes. Hence, any $\delta_i \in [\underline{\delta}, \bar{\delta}]$ is an equilibrium strategy.

Now consider lower initial reputation levels.

Following Lemma defines two-periods equilibrium.

Lemma 5. *If $n = 2$, following assessment constitutes the agent-optimal PBE.* $\mu_1 = 0 = \nu_1$

$$\delta_1 \in \begin{cases} [\underline{\delta}, \bar{\delta}], & \rho_1 \geq 1-2k \\ \delta_1' = \dfrac{1-2k-\rho_1}{1-2k-\rho_1+\rho_1 k}, & \dfrac{1}{2} - k \leq \rho_1 < 1-2k \end{cases}$$

$$\lambda_1 = \begin{cases} 0, & \rho_1 < \dfrac{1}{2} - k \\ 1, & \rho_1 \geq \dfrac{1}{2} - k \end{cases}$$

Proof. Lemma 5

Again, $\rho_1 \geq 1-2k$ is obvious. Consider $\rho_1 < 1-2k$. Note that $\rho_2(1) < 1-2k$ in any equilibrium, hence $\lambda_2(1) = 0$. There are two possibilities:

Case 1

Agent has no reputation concern in the first period. In such case, V_1^g is decreasing in δ_1. Hence, the agent chooses minimum possible δ_1 so that the principal gets a non-negative payoff.

$$V_1^P = (1-\delta_1)\left(k - \frac{1}{2} + \frac{1}{2}\rho_1\right) + \frac{1}{2}\rho_1 k \delta_1$$

V_1^P is non-negative if $\delta_1 \geq \delta_1^{'} := \dfrac{1-2k-\rho_1}{1-2k-\rho_1+\rho_1 k}$ Thus, if there is no reputation concern in the lowest range of initial reputation so that the agent is hired, $\delta_1 = \delta_1^{'}$ as the payoff of the agent is decreasing in δ_1.

$\delta_1^{'} \leq \dfrac{1}{1+k}$ holds as long as $\rho_1 \geq \dfrac{1}{2} - k$. Thus, the agent is not hired for $\rho_1 < \dfrac{1}{2} - k$, if the agent has no reputation concern.

Case 2

Agent has reputation concern in the first period. Note that $\nu_1 = 1$ can not be an equilibrium strategy, because it implies $\rho_2(0) = \rho_1 < 1-2k$ by Bayes' rule and $\lambda_2(0) = 0$. Thus, $\nu_1 < 1$. Only δ_1 value satisfying this is $\dfrac{1}{1+k}$. Principal's payoff:

$$V_1^P = \frac{k}{1+k}\left(k - \frac{1}{2} + \frac{1}{2}\rho_1\right) + \frac{1}{1+k}\frac{1}{2}\rho_1 k$$

which is non-negative if $\rho_1 \geq \dfrac{1}{2} - k$. Hence, in any case, the agent is not hired if $\rho_1 < \dfrac{1}{2} - k$.

Note that:

$$V_1^{g[case1]} = (1-\delta_1^{'})k + \frac{1}{2}\delta_1^{'} k$$

$$V_1^{g[case2]} = \frac{k^2 + \frac{1}{2}k}{1+k}$$

and $\delta_1^{'} \leq \dfrac{1}{1+k}$ implies $V_1^{g[case1]} \geq V_1^{g[case2]}$, which holds strictly if $\delta_1^{'} < \dfrac{1}{1+k}$. Thus, no reputation concern setup is preferred by the agent. □

I will prove the rest with induction method. Consider period $n-2$. If, $\rho_{n-2} \geq 1-2k$, $\delta_{n-2} = \underline{\delta}$. Consider $\rho_{n-2} < 1-2k$. From Lemma 5, $V_{n-1}^P = 0$ if $\rho_{n-1} < 1-2k$. As $\rho_{n-2} < 1-2k$ implies $\rho_{n-1}(1) < 1-2k$, $V_{n-1}^P(|a_n - 2 = 1) = 0$.

Consider two cases:

Case 1

Agents sets δ_{n-2} so that there is no reputation concern.

$$V^P_{n-2} = (1-\delta_{n-2})\left(k - \frac{1}{2} + \frac{1}{2}\rho_{n-2}\right) + \delta_{n-2}\frac{1}{2}\rho_{n-2}k$$

Thus, the agent sets minimum δ_{n-2} providing non-negative payoff to the principals, i.e. $\delta_{n-2} = \delta^{'}_{n-2}$. Agent is not hired if $\rho_{n-2} < \frac{1}{2} - k$.

Case 2

Agent sets δ_{n-2} so that there are reputation concerns.

1. $\rho_{n-1}(0) \geq \frac{1-2k}{1-k}$. This implies that if action 0 is observed in period $n-2$, the agent will be hired for the rest of the game irrespective of a_{n-1}. Define $\tilde{\delta}_{n-2}$ which makes the agent indifferent. If $\mu_{n-2} = 1$, then $\rho_{n-1}(1) = 0$. In this case, $\lambda_{n-1}(1) = 0$ and $\tilde{\delta}_{n-2} = \frac{1}{1+k}$. Thus, the problem is the same with two periods interaction. I showed in Lemma 5 that the agent prefers no reputation setup. Hence, $\mu_{n-2} < 1$. Good agent is either indifferent or strictly prefers action 1. Hence, for the sake of exposition, assume $\mu_{n-2} = 0$, wlog.

$$V^{g[rep]}_{n-2} = (1-\tilde{\delta}_{n-2})k + \frac{1}{2}\tilde{\delta}_{n-2}k + \frac{1}{2}\tilde{\delta}_{n-2}V^{g[rep]}_{n-1}(|a_{n-2} = 1)$$

where $V^{g[rep]}_{n-1}(|a_{n-2} = 1) = 0$ if $\rho_{n-1} < \frac{1}{2} - k$ and

$$(1-\delta^{'}_{n-1})k + \frac{1}{2}\delta^{'}_{n-1}k$$

otherwise.

On the other hand, if the agent sets $\delta_{n-2} = \delta^{'}_{n-2}$ as in case 1, she gets:

$$V^{g[norep]}_{n-2} = (1-\delta^{'}_{n-2})k + \frac{1}{2}\delta^{'}_{n-2}k + \frac{1}{2}\delta^{'}_{n-2}V^{g[norep]}_{n-1}(|a_{n-2} = 1)$$

Note that, $\delta^{'}_{n-2} \leq \tilde{\delta}_{n-2}$ and $V^{g[norep]}_{n-1}(|a_{n-2} = 1) \geq V^{g[rep]}_{n-1}(|a_{n-2} = 1)$ (as $\rho_{n-1}(1)$ decreases with μ_{n-2}). Hence, agent prefers to set $\delta_{n-2}^{'}$.

2. Hence, if there are reputation concerns in the equilibrium in period $n-2$, it must be so that $\frac{1}{2}-k \leq \rho_{n-1}(0) < 1-2k$. I need $\nu_{n-2} \in (0,1)$ by Bayes' rule. Then $\mu_{n-2}=1$. Note that, if $\mu_{n-2}=1$, $\rho_{n-1}(1)=0$ and $\lambda_{n-1}(1)=0$. Hence, δ_{n-2} is such that the bad agent is indifferent:

$$(1-\delta_{n-2})k = (1-\delta_{n-2})(k-1)+\delta_{n-2}(1-\underline{\delta})k$$

which implies:

$$\delta_{n-2} = \frac{1}{1+k-\underline{\delta}}$$

The good agent's payoff is given by:

$$V_{n-2}^{g[case2]} = \frac{k-\underline{\delta}k}{1+k-\underline{\delta}k}\left(k-\frac{1}{2}\right)+\frac{1}{1+k-\underline{\delta}k}\left(k-\frac{1}{2}\underline{\delta}k\right)$$

The good agent's payoff in case 1 where there are no reputation concerns:

$$V_{n-2}^{g[case1]} = (1-\delta_{n-2}^{'})k+\frac{1}{2}\delta_{n-2}^{'}k+\frac{1}{2}\delta_{n-2}^{'}V_{n-1}^{g}(|a_{n-2}=1)$$

note that $V_{n-1}^{g}(|a_{n-2}=1)<k$, hence $V_{n-2}^{g[case1]}$ is decreasing in $\delta_{n-2}^{'}$. It implies that $V_{n-2}^{g[case1]} > \frac{k^2+\frac{1}{2}k}{1+k} > V_{n-2}^{g[case2]}$. Hence the agent prefers to set δ_{n-2} to the lowest possible value and there are no reputation concerns.

Finally, consider period i such that strategies are as defined above in periods $i+1,...n$. If, $\rho_i \geq 1-2k$, $\delta_i = \underline{\delta}$. Consider $\rho_i < 1-2k$. From above analysis, $V_{i+1}^{P}=0$ if $\rho_{i+1}<1-2k$. $\rho_i < 1-2k$ implies $\rho_{i+1}(1)<1-2k$, $V_{i+1}^{P}(1)=0$. So, following the same steps above, if agent sets δ_i so that there are no reputation concerns, then $\delta_i=\delta_i^{'}$.

Now, suppose the agent sets δ_i so that there are reputation concerns. Define $s,t \in \{1,2,...,n-i\}$ such that $s<t$. Let t represent the number of periods the agent is hired if action 1 is observed in all t periods following period $a_i=0$. Following the reasoning above, the agent will not set $t=n-i$. Hence, if there are reputation concerns, then $t<n-i$. It implies good agent has higher reputation concerns, hence $\mu_i=1$ and $\nu_i \in (0,1)$. Note that, t represents the number of periods after which

the reputation falls below $\frac{1}{2}-k$ if no action 0 is observed. Let s represent the number of periods after which the reputation falls below $1-2k$. The reason for making this restriction is that in any period the agent's reputation fall below $1-2k$, the agent can no more set $\delta_j=\underline{\delta}$ but rather $\delta_j=\delta_j^{'}$.

Define Z as the payoff of the agent in these t periods. Note that, both type of agent get a payoff of k in each of these periods. Then Z has the following form:

$$Z:=\left(1-\underline{\delta}^{t-s}\right)k+\underline{\delta}^{t-s}\tilde{V}$$

where $\tilde{V}$ stands for the payoff in the last s periods. $\tilde{V}<k$ as agent sets $\delta_j=\delta_j^{'}$ in these periods. Hence, $Z\leq k$. In order to find the form of agent's payoff, I first need to find δ_i. Note that, $v_i\in(0,1)$, hence, the sequential rationality constraint of the bad agent:

$$(1-\delta_i)k=(1-\delta_i)(k-1)+\delta_i Z$$

hence, $\delta_i=\frac{1}{1+Z}$.

Payoff of the good agent becomes:

$$V_i^{g[case2]}=\frac{Z}{1+Z}\left(k-\frac{1}{2}\right)+\frac{1}{2^t}\frac{Z}{1+Z}+\left(1-\frac{1}{2^t}\right)\frac{k}{1+Z}$$

where $\frac{1}{2^t}$ is the probability that no action 0 is observed in t periods. With the complementary probability, the agent reveals herself as the good type and gets payoff of k. $V_i^{g[case2]}$ is continuous and monotonically increases in $\underline{\delta}$. Moreover, it is equal to $\frac{k^2+1/2k}{1+k}$ if $\underline{\delta}=0$. Recall that $V_{n-2}^{g[case1]}>\frac{k^2+1/2k}{1+k}$, hence, if $\underline{\delta}$ is sufficiently small, then the good agent prefers to set stakes so that there are no reputation concerns. □

Proof. Proposition 6

Suppose there is an equilibrium where $\frac{2^{t+1}(1-2k)}{2^{t+1}(1-2k)+k}\leq\rho_1$ and the principal sets $\delta_1<\frac{1}{1+k}$.

$$V_1^P=(1-\delta_1)\left(k-\frac{1}{2}+\frac{1}{2}\rho_1\right)+\delta_1 V_2^P$$

By Bayes' rule, the agent is hired for at leas $t+2$ periods in the equilibrium. In any following period, the principal can set δ values so that the agent has no reputation concern in any period. Thus,

$$V_2^P \geq \frac{1}{2}\rho_1 k + \frac{1}{2}\left(1 - \frac{1}{2^{t+1}}\right)\rho_1 k$$

which is greater than $k - \frac{1}{2} + \frac{1}{2}\rho_1$, by $\rho_1 < \frac{2^{t+1}(1-2k)}{2^{t+1}(1-2k)+k}$. Hence, V_1^P is increasing in δ_1. Thus, the principal will deviate to a higher δ_1. Hence, in any equilibrium, $\delta_1 \geq \frac{1}{1+k}$. The same reasoning applies to period i. □

Proof. Proposition **??**

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□

Proof. Proposition 7

The following Lemma will be used in the proof.

Lemma 6. *If $\rho_1 \leq \frac{(1-2k)2^{n-2}}{(1-2k)2^{n-2}+k}$, then there is an evaluation phase in any equilibrium i.e. $\lambda_i(0) = 1$ and $\lambda_i(1) = 0$ for some $i \in N$, and the strategy of the agent in the evaluation phase is as following: $\mu_i = 1$ and $\nu_i \in [0,1]$. The last period of the evaluation phase may be an exception where $\mu_i \in [0,1]$ and $\nu_i = 0$ may hold.*

Proof. Lemma 6.

Suppose that there is an equilibrium where there is no evaluation phase. If $\lambda_i(0) = 1$ and $\lambda_i(1) = 0$ for all $i \in N$ in the equilibrium, then the agent has no incentive to play action 0 with a positive probability: $\mu_i = 0$ and $\nu_i = 0$ for all $i \in N$. By Bayes' rule, $\rho_1 \leq \frac{(1-2k)2^{n-2}}{(1-2k)2^{n-2}+k}$ implies $\rho_n(1) < 1-2k$ if $a_i = 1$ is observed in each period $i \in N/n$. Thus, $\lambda_n(1) = 0$, contradiction.

Next, I will show that $\mu_i = 1$ and $\nu_i \in [0,1]$ in the evaluation phase. In any period where action 0 does not lead to the agent being hired for all the remaining periods, then the good agent has strictly higher reputational concerns than the bad agent. Fix a period i so that $\lambda_{i+1}(0) = 1$ and $\lambda_{i+1}(1) = 0$. Then the expected continuation payoff of the good agent is higher than that of bad agent in any such period i. Consider the following: the bad agent can improve her reputation by playing the action 0, but it costs her today's payoff. So she sacrifices that period's gain in order to get more chance for further periods. On the other hand, suppose $\theta_i = 1$ is observed and the good agent plays $a_i = 0$ in period i. She sacrifices period i payoff but in return she gets more chance to play her favorite action in the future. In addition, the good agent gets more chance to improve her reputation costlessly in the future (when the true state is 0 and she plays accordingly). Consider the following. Suppose the evaluation phase also involves the period $i+1$. In period $i+1$, the expected payoff of the good agent from playing action 0 is given by $k - prob\left[\theta_{i+1} = 1\right]1 - prob\left[\theta_{i+1} = 0\right]0 = k - \frac{1}{2}$. Whereas, the expected payoff of the bad type from playing action 0 in period $i+1$ is given by $k - prob\left[\theta_{i+1} = 1\right]1 - prob\left[\theta_{i+1} = 0\right]1 = k - 1$. Thus, building reputation is costlier for the bad agent in period i. If period i is not the last period of the evaluation phase, the good agent's expected continuation payoff is always higher than that of the bad agent. If period i is the last period of the evaluation phase so that there is a perfect screening in the period, then the future expected payoff is the same for both types.

Suppose $\nu_i = 0$, then $\rho_{i+1}(0) = 1$. Thus, if $\nu_i = 0$, then the period i serves as the evaluation period, and the agent is hired until the end if action 0 is observed in period i. It is possible in the last period of the evaluation phase and $\mu_i \in [0,1]$ satisfies the equilibrium condition. The reasoning is that the last period of the evaluation phase, the decision is exactly same with the two-periods game. If i is not the last period of the evaluation phase, i.e. $\lambda_{i+1}(0) = 1$ and $\lambda_{i+1}(1) = 0$, then $\nu_i > 0$.

Suppose $\nu_i = 1$, it implies $\mu_i = 1$. Then there will be no update on the type of the agent and $\rho_{i+1}(0) = \rho_i$. The reputation is not updated and the principal gets a negative expected stage game payoff in period i. As $\delta_i < 1$, $\mu_i = 1 = \nu_i$ can not constitute an equilibrium strategy. Hence, I need to have $\nu_i \in (0,1)$ in the evaluation phase may be except from the last period. From the reasoning above, the good agent's future expected continuation payoff is larger than that of the bad agent. For the δ_i level that the bad agent is indifferent, the good agent strictly prefers playing action 0.

Hence, if $\rho_1 \leq \frac{(1-2k)2^{n-2}}{(1-2k)2^{n-2}+k}$, then there will be evaluation phase in any equilibria and the equilibrium strategy of the agent in the evaluation phase is given by $\mu_i = 1$ and $\nu_i \in (0,1)$ maybe expect from the last period. □

Consider the following strategies:

$$\nu_i \in \begin{cases} \{0\}, & \delta_i < \frac{1}{1+k} \\ [0,1], & \frac{1}{1+k} \leq \delta_i \leq \frac{\sum_{h=0}^{n-i-1} k^h}{\sum_{h=0}^{n-i} k^h} \\ \{1\}, & \delta_i > \frac{\sum_{h=0}^{n-i-1} k^h}{\sum_{h=0}^{n-i} k^h} \end{cases},$$

$$\mu_i \in \begin{cases} \{0\}, & \delta_i < \frac{1}{1+k} \\ [0,1], & \frac{1}{1+k} \leq \delta_i \leq \delta_i^g \\ \{1\} & \delta_i > \delta_i^g \end{cases} \text{ such that } \frac{1}{1+k} \leq \delta_i^g \leq \frac{\sum_{h=0}^{n-i-1} k^h}{\sum_{h=0}^{n-i} k^h} \text{ for } i \in N/n,\ \mu_n = 0\ ,\nu_n = 0.$$

Suppose that the initial reputation level ρ_1 is close to 0. There will be an evaluation phase from Lemma 6. If ρ_1 is the smallest initial reputation so that the project is run, then the evaluation phase will start in the first period. The reasoning is that, if $\lambda_2(0) = 1 = \lambda_2(1)$, then $\rho_2(1) < \rho_1$. As ρ_1 is the smallest reputation level for the project to be implemented, $\rho_2(1) < \rho_1$ implies $\lambda_2(1) = 0$, a contradiction. Suppose that the first $m-1$ periods serve as the evaluation phase. Then $\lambda_m(0) = 1$ and $\lambda_m(1) = 0$ and $\lambda_{m+1}(0) = 1$ and $\lambda_{m+1}(1) = 1$. In the evaluation phase, the strategy of the agent: $\mu_i = 1$ and $\nu_i \in (0,1)$ may be except from the last period. Thus, in each period of the evaluation phase the principal gets an expected payoff of $k - \frac{1}{2}$. Define $q_i := prob[a_i = 0] = \rho_i (1 + \mu_i) + (1 - \rho_i) \nu_i$. The expected continuation payoff of the principal is given as follows:

$$\begin{aligned} V_1^P = \quad & \gamma_1 \left(k - \frac{1}{2}\right) + \gamma_2 q_1 \left(k - \frac{1}{2}\right) + \gamma_3 q_1 q_2 \left(k - \frac{1}{2}\right) + \ldots \\ & \ldots + \gamma_{m-1} \prod_{i=1}^{m-2} q_i \left(k - \frac{1}{2}\right) + \delta_m \prod_{i=1}^{m-1} q_i V_m^P \end{aligned}$$

Note that V_1^P is increasing in ρ_1 and $V_1^P = \gamma_1 \left(k - \frac{1}{2}\right)$ if $\rho_1 = 0$.

If the period $m-1$ is the last period of the evaluation phase, then the sequential rationality constraint of the bad agent in period $m-1$ is given as:

$$(1-\delta_{m-1})\,k = \delta_{m-1}\,(k-1) + (1-\delta_{m-1})\,f\,(k)$$

where $f\,(k) = V_m^B$ is increasing function of k such that $f\,(k) \le k$. The sequential rationality constraint implies $\delta_{m-1} = \frac{1}{1+f\,(k)}$. The sequential rationality constraint of the bad agent in period $m-2$ is given as:

$$(1-\delta_{m-2})\,k = \delta_{m-2}\,(k-1) + (1-\delta_{m-2})\,\frac{f\,(k)\,k}{1+f\,(k)}$$

hence $\delta_{m-2} = \frac{1+f\,(k)}{1+f\,(k)+f\,(k)\,k}$. Iterating in the same fashion, $\delta_i = \frac{1+\sum\limits_{h=1}^{m-i-1} f(k)k^{h-1}}{1+\sum\limits_{h=1}^{m-i} f(k)k^{h-1}}$ for all $i \in \{0,1,...,m-2\}$ and $\delta_{m-1} = \frac{1}{1+f\,(k)}$. Thus, $\delta_1 = \frac{1+\sum\limits_{h=1}^{m-2} f(k)k^{h-1}}{1+\sum\limits_{h=1}^{m-1} f(k)k^{h-1}}$. By definition, $\gamma_1 = 1-\delta_1 = \frac{f(k)k^{m-2}}{1+\sum\limits_{h=1}^{m-1} f(k)k^{h-1}}$. Given $f\,(k) \le k < \frac{1}{2}$, γ_1 is decreasing in m. $\left(k-\frac{1}{2}\right)$ is decreasing in γ_1 and is maximized when $m=n$. Increasing the span of the evaluation phase increases the expected continuation payoff of the principal if the initial reputation is close enough to 0. Hence, in the principal optimal equilibrium, the evaluation phase includes all the periods except period n.

Note that, if $m=n$, then $f(k)=k$, hence $\delta_i^* = \frac{1+\sum\limits_{h=1}^{n-i-1} f(k)k^{h-1}}{1+\sum\limits_{h=1}^{n-i} f(k)k^{h-1}}$ and $\lambda_i^*(0)=1$ and $\lambda_i^*(1)=0$ for all $i \in N$.

Note that $\gamma_i = (1-\delta_i)\prod\limits_{j}^{i-1}\delta_j$ thus,

$$\gamma_i = \frac{1-k}{1-k^n}k^{n-i}$$

The last thing to show is that there exist small enough initial reputation level that provides a positive expected continuation payoff to the principal for the above-defined strategies.

The expected continuation payoff of the principal is given below:

$$V^{P*} = \gamma_1\left(k-\frac{1}{2}\right) + \gamma_2 q_1\left(k-\frac{1}{2}\right) + \gamma_3 q_1 q_2\left(k-\frac{1}{2}\right) + \dots$$
$$\dots + \gamma_{n-1}\prod_{i=1}^{n-2} q_i\left(k-\frac{1}{2}\right) + \gamma_n\prod_{i=1}^{n-1} q_i\left(k-\frac{1}{2}+\frac{1}{2}\rho_n\right)$$

Note that the highest ρ_{n-1} satisfying the equilibrium conditions is $\frac{1-2k}{1-k}$. Hence, in the principal optimal equilibrium, $v_{n-2} = \frac{k\rho_{n-2}}{(1-2k)(1-\rho_{n-2})}$. I also know from the two-periods analysis that $v_{n-1} = 0$. For other v_i values, the principal's utility is decreasing in v_i. Thus, the expected continuation payoff of the principal will be higher than the case where $v_i = 1$ for all $i \in N/\{n-2, n-1, n\}$. Define the expected continuation payoff of the principal as $\underline{V}^{P*}$ as the lower bound of the payoff he will get, which holds when $v_i = 1$ for all $i \in N/\{n-2, n-1, n\}$.

Simplify the payoff:

$$\underline{V}^{P*} = \frac{1-k}{1-k^n}\left(k^{n-1}\left(k-\frac{1}{2}\right) + k^{n-2}\left(k-\frac{1}{2}\right) + k^{n-3}\left(k-\frac{1}{2}\right) + \dots\right.$$
$$\left.\dots + k^2\left(k-\frac{1}{2}\right) + k\left(\rho_1 + (1-\rho_1)\frac{k\rho_1}{(1-2k)(1-\rho_1)}\right)\left(k-\frac{1}{2}\right) + \rho_1 k\right)$$

where I used the fact that, if $v_i = 1$ for all $i \in N/\{n-2, n-1, n\}$, there is no update on the beliefs until the last two periods. Thus, $prob\,[a_{n-2} = 0] = \rho_{n-2} + (1-\rho_{n-2})\,v_{n-2} = \rho_1 + (1-\rho_1)\frac{k\rho_1}{(1-2k)(1-\rho_1)}$. Moreover, as the principal's payoff does not depend on μ_{n-1}, I assume $\mu_{n-1} = 1$ for the sake of simplicity. Then, $prob[a_{n-1} = 0] = \rho_{n-1} + (1-\rho_{n-1})\,v_{n-1} = \frac{\rho_1}{\rho_1 + (1-\rho_1)\,v_{n-2}}$ and $\rho_n = 1$ as $v_{n-1} = 0$. The payoff becomes:

$$\underline{V}^{P*} = \frac{1-k}{1-k^n}\left\{\frac{k^2-k^n}{1-k}\left(k-\frac{1}{2}\right)+\frac{1}{2}\rho_1 k\,(1+k)\right\}$$

which is non-negative for: $\rho_1 \geq \frac{(1-2k)\left(k-k^{n-1}\right)}{(1-k^2)}$. Hence there exist such ρ_1 where V^{P*} is non-negative. □

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