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Anonymous and Neutral Social Choice: Existence Results on Resoluteness

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Abstract

We present the existence conditions of anonymous, neutral and resolute social welfare and social choice rules in a group theoretical framework. We define the maximum domain at which such aggregation rules exist. We propose a tie-breaking procedure to obtain resolute refinements of social choice rules, which preserves anonymity and neutrality. The conditions for this refinement to satisfy simple monotonicity are compared with such conditions for the resolute refinements obtained via conventional tie-breaking mechanisms.

keywords: anonymous, neutral and resolute social welfare/choice rules, group theory, anonymous and neutral tie-breaking, monotonicity.

1 Introduction

Anonymity and neutrality are the most basic requirements to meet the equality principle of democracy. Anonymity demands a social aggregation mechanism to treat each voter's opinion equally whereas neutrality requires no outcome to be a priori favored. It is one of the major impossibility results in social choice theory that there is no anonymous and neutral mechanism that aggregates individual preferences into resolute social decisions. However, resoluteness is an important criterion since, in numerous real-life cases (such as the presidential elections, referendums, legislative voting, etc.), only a single social decision is to be adopted by a society.

Moulin (1983) is the first study that presents the existence conditions for anonymous, neutral and resolute social choice rules. For m alternatives linearly ranked by n voters, it shows that an anonymous, neutral and resolute (ANR) social choice rule exists if and only if m cannot be written as the sum of non-trivial divisors of n. Based on the group theoretical framework developed by Eğecioğlu and Giritligil (2013), we re-examine this result and extend the analysis to the existence of ANR social welfare rules. We prove that there exists an ANR social welfare rule if and only if the greatest common divisor of $m!$ and n is one, i.e., $gcd(m!, n) = 1$.[1] Our results indicate that this condition characterizes more demanding social choice rules, i.e., social choice rules which are not only anonymous, neutral and resolute but also paretian (respecting the unanimous preferences of voters over pairs of alternatives). Campbell and Kelly (2015) focuses on how dramatic the departures from pareto optimality are, when this common divisor condition is not met.[2] Regarding ANR social choice rules which are not necessarily paretian, we confirm the characterization by Moulin (1983) through our group theoretical approach. We analyze how restrictive the conditions for the existence of ANR social choice and social welfare rules are.

The framework through which we study the aforementioned existence conditions is based on the group theoretical formulation of equivalence classes which are defined, for given m and n, on the set of preference profiles as partitions induced by permutations defined on the set of voters and alternatives. This framework enables us to characterize those equivalence classes which yield cyclical preference structures leading to unavoidable multiplicity of outcomes by anonymous and neutral social aggregation rules. That is, when the existence conditions are not met, the group theoretical formulation enables us to identify the "problematic" equivalence classes and to introduce the maximum domain of preference profiles on which ANR social welfare and social choice rules can be defined.

When a voting takes place in order to choose only one alternative as the winner and the adopted social choice rule remains indecisive between two or more alternatives, a procedure is needed to break the tie(s) in order to choose only one of these tied alternatives. There are two conventional tie-breaking procedures. The *tie-breaking agenda* which is proposed by Moulin (1988a, 1988b)

[1] Based on the framework introduced by Eğecioğlu and Giritligil (2013), Bubboloni and Gori (2014) also provides this result.

[2] Campbell and Kelly (2015) shows that when m is greater than the smallest prime dividing n, all anonymous, neutral and resolute social choice rules sometimes choose pareto inferior alternatives as winners.

breaks ties using a predetermined strict ordering of the alternatives. The *tie-breaking voter* procedure, on the other hand, breaks ties according to the relative positions of the tied alternatives in a predesignated voter's preference ranking. Clearly, the resolute refinements of a social choice rule obtained via the former procedure violate neutrality whereas the ones obtained via the latter violate anonymity. Based on our characterization of ANR social welfare functions, we propose a tie-breaking procedure to obtain resolute refinements of anonymous and neutral social choice rules while preserving their anonymity and neutrality.

As the final task of this paper, we examine whether the resolute refinements of anonymous and neutral social choice rules that are obtained via these three tie-breaking procedures satisfy simple monotonicity (requiring an alternative to remain as a winner when its position is improved upon every voter's preference). Our results show that the resolute refinement of an anonymous and neutral social choice rule obtained through the tie-breaking agenda satisfies simple monotonicity if and only if the rule itself is exclusively monotonic (requiring an alternative to remain as winner when its position is improved upon every voter's preferences and at the same time, demanding that such a transition does not allow any other alternative to become a winner if it was not in the first place). We show that exclusive monotonicity is not a necessary but a sufficient condition for the resolute refinement obtained through the tie-breaking voter procedure to be simple monotonic. Moreover, we prove that some of the ANR refinements (obtained through our tie-breaking procedure) of exclusively monotonic social choice rules satisfy simple monotonicity whereas all ANR refinements of a social choice rule are simple monotonic if and only if the rule itself is sensitively monotonic (requiring an alternative to become the unique winner when its position is improved upon every voter's preference).

The rest of paper is organized as follows. In Section 2, we present our basic definitions and group theoretical framework. Our results on the characterizations of ANR social welfare and social choice rules are given in Section 3. This section also contains our definition of maximal domains for such rules. In Section 4, we introduce a tie-breaking rule to obtain anonymous and neutral resolute refinements of social choice rules. Considering this and two conventional tie-breaking rules, we examine the conditions for resolute refinements to satisfy simple monotonicity principle. We present our concluding remarks in Section 5.

2 Framework

2.1 Basic Notations and Definitions

We consider a finite group of *voters*, $N = \{1, 2, ..., i, j, ..., n\}$, who wish to collectively choose from a finite set $M = \{a, b, c, ...\}$ of $m \in \mathbb{N}$ *alternatives*. Voters have preferences that are complete, transitive and antisymmetric binary relations on M and voter i's preference is denoted by p^i. Thus, for any two alternatives $a, b \in M$, ap^ib means that voter i strictly prefers a to b. The set of all complete and transitive binary relations (a.k.a. weak orders) on M is denoted by $\mathcal{W}(M)$, whereas the set of all complete, transitive and antisymmetric binary relations (a.k.a. linear orders) on M is denoted by $\mathcal{L}(M)$. A member $\pi = (p^i)_{i \in N}$ of $\mathcal{L}(M)^N$ is called a *preference profile (or profile)*. For any profile $\pi = (p^i)_{i \in N}$ and any non-empty subset M' of M, the *projection of* π *onto* M' is denoted by $\pi_{|M'} = (p^i_{|M'})_{i \in N} \in \mathcal{L}(M')^N$ and defined by: $\forall i \in N, \forall a, b \in M', ap^ib \Leftrightarrow ap^i_{|M'}b$.

A *social welfare rule* (SWR) $f : \mathcal{L}(M)^N \to \mathcal{W}(M)$ is a function that assigns a weak order to each preference profile. We call an SWR *resolute* if it assigns a linear order to every preference profile, i.e., $f(\pi) \in \mathcal{L}(M)$ for all $\pi \in \mathcal{L}(M)^N$. A resolute SWR is called a *social welfare function* (SWF). A *social choice rule* (SCR), $F : \mathcal{L}(M)^N \to 2^M \setminus \{\emptyset\}$, is a mechanism that returns, for each profile $\pi \in \mathcal{L}(M)^N$, a nonempty set of alternatives $F(\pi) \subseteq M$. A SCR is *resolute* if its outcome is a unique alternative for each profile, i.e., $|F(\pi)| = 1$ for all $\pi \in \mathcal{L}(M)^N$. A resolute SCR is called a *social choice function* (SCF).

A *permutation* on the finite set N is a bijection on N. The set of all permutations on N is denoted by S_N. When a preference profile $\pi \in \mathcal{L}(M)^N$ is rearranged according to permutation $\eta \in S_N$, the resulting preference profile is denoted by π^η. The way that the elements of N are permuted by η can be represented by a cycle decomposition of η. For $n = 3$, for example, the cycle decomposition of $\eta = (12)(3)$ means that η appoints the first voter as the second and the second voter as the first, and it fixes the third voter. A permutation on M is defined similarly and the set of all permutations on M is denoted by S_M. $\alpha(\pi)$ denotes the rearrangement of π according to permutation $\alpha \in S_M$. Note that $\forall \pi, \tilde{\pi} \in \mathcal{L}(M)^N, \forall \alpha, \lambda \in S_M$ and $\forall \eta \in S_N$, these operations satisfy:

- Associativity: $\alpha(\pi)^\eta = [\alpha(\pi)]^\eta = \alpha(\pi^\eta)$.
- Cancellation law: $\pi^\eta = \tilde{\pi} \Rightarrow \pi = \tilde{\pi}^{\eta^{-1}}$ and $\alpha(\pi) = \tilde{\pi} \Rightarrow \pi = \alpha^{-1}(\tilde{\pi})$.
- Injectivity: $\alpha \neq \lambda \Rightarrow \alpha(\pi) \neq \lambda(\pi)$.[3]

Definition 2.1. *A SWR f is* ***anonymous*** *if it treats each voter equally, i.e., $\forall \eta \in S_N, \forall \pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N, f(\pi) = f(\pi^\eta)$ where $\pi^\eta = (p^{\eta(i)})_{i \in N}$.*

Definition 2.2. *A SWR f is* ***neutral*** *if it treats each alternative equally, i.e., $\forall \alpha \in S_M, \forall \pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N, f(\alpha(\pi)) = \alpha(f(\pi))$ where $\alpha(\pi) = (\alpha(p^i))_{i \in N}$ is the preference profile defined as $\forall a, b \in M, \forall i \in N, ap^ib \Leftrightarrow \alpha(a)\alpha(p^i)\alpha(b)$.*

Definition 2.3. *A SCR F is* ***anonymous*** *if $\forall \eta \in S_N, \forall \pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N, F(\pi) = F(\pi^\eta)$ where π^η is defined as above.*

Definition 2.4. *A SCR F is* ***neutral*** *if $\forall \alpha \in S_M, \forall \pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N, F(\alpha(\pi)) = \alpha(F(\pi))$ where $\alpha(\pi)$ is defined as above.*

[3]Note that injectivity is not satisfied for permutations on voters, i.e., $\forall \eta \in S_N, \exists \gamma \in S_N, \exists \pi \in \mathcal{L}(M)^N$ such that $\eta \neq \gamma$ and $\pi^\eta = \pi^\gamma$.

2.2 Group Theoretical Formulation

Group theory provides a very convenient framework to study the existence of social aggregation procedures that satisfy both anonymity and neutrality since these properties are based on permutations on N and M, respectively. We commence by giving a formal definition of a group.

Definition 2.5. *A set G, together with an operation $\bullet : G \times G \rightarrow G$, is called a **group** if the following conditions hold:*

i) $\forall x, y, z \in G, x \bullet (y \bullet z) = (x \bullet y) \bullet z$;

*ii) There exists an element of G, called the **identity** and denoted by I_G, such that $\forall x \in G, I_G \bullet x = x = x \bullet I_G$;*

iii) $\forall x \in G, \exists x^{-1} \in G$ *such that* $x \bullet x^{-1} = I_G = x^{-1} \bullet x$.

It should be clear from this definition that both S_N and S_M, together with the composition operation, are *symmetric groups on* N and M, respectively.

Definition 2.6. *A group $(G, \bullet)$ is said to **act** on a set X if $\forall g \in G, \exists \theta_g : X \rightarrow X$ such that*

i) $\forall g, h \in G, \theta_g \circ \theta_h = \theta_{g \bullet h}$;

ii) $\forall x \in X, \theta_{I_G}(x) = x$.

Let the symmetric group S_N act on $\mathcal{L}(M)^N$ via the functions $\theta_\eta : \mathcal{L}(M)^N \rightarrow \mathcal{L}(M)^N$ defined by $\forall \pi \in \mathcal{L}(M)^N, \forall \eta \in S_N, \theta_\eta(\pi) = \pi^\eta$. Obviously, this family of functions satisfies the conditions stated in definition 2.6.

Definition 2.7. *Let a group $(G, \bullet)$ act on a set X. For every $x \in X$, the **orbit** of x, denoted by $O(x)$, is the subset of X that is defined by $O(x) = \{\theta_g(x) | g \in G\}$.*

Proposition 2.1. *Let a group $(G, \bullet)$ act on a set X. Then, X is the disjoint union of the orbits.*

We state the proposition without proof as the interested reader can find it in almost any abstract algebra textbook. When we consider S_N acting on $\mathcal{L}(M)^N$ via the functions θ_η described above, the orbit of a preference profile $\pi \in \mathcal{L}(M)^N$ is the set $\{\pi^\eta | \eta \in S_N\}$. In other words, the orbit of a preference profile π is the set of all profiles that can be obtained via permuting the names of voters. Note that, by definition, each orbit is nonempty and orbits of two different preference profiles either coincide or are disjoint. This implies that group action of S_N on $\mathcal{L}(M)^N$ partitions $\mathcal{L}(M)^N$ into nonempty orbits $O_1, O_2, ..., O_K$ and we can write $\mathcal{L}(M)^N = O_1 + O_2 + ... + O_K$ where "+" denotes disjoint union of sets. We will call each orbit an *anonymous equivalence class* (AEC) since each orbit is the *class* of all preference profiles that are *equivalent* when the voters are viewed *anonymously*.

Each AEC can be represented with a *composition vector* $v \in \mathbb{N}^{\mathcal{L}(M)}$, each entry of which denotes the number of times the corresponding preference ordering is adopted by the electorate. Note that, as $|\mathcal{L}(M)| = m!$, each composition vector has $m!$ entries and the sum of those entries must be equal to the number of voters. On the other hand, each vector of $m!$ components, each of which is a natural number and the sum of which is n corresponds to an AEC. The set of all AECs will be denoted by $V(M,N) = \{v : \mathcal{L}(M) \to \mathbb{N} \text{ and } \sum_{p \in \mathcal{L}(M)} v(p) = n\} = \{O_1, O_2, ..., O_K\}$. For the rest of this paper, we adopt v both to denote a composition vector and the AEC represented by it. We use $v(p)$ if v is a composition vector. On the other hand, $\pi \in v$ means that v is an AEC and π is one of the preference profiles in that AEC.

Now let S_M act on $V(M,N)$. Then, the family of functions that definition 2.6 requires is given as follows: For all $\alpha \in S_M$, define $\theta_\alpha : V(M,N) \to V(M,N)$ by: $[\theta_\alpha(v)](p) = v(\alpha(p))$ for every $v \in V(M,N)$ and $p \in \mathcal{L}(M)$. These functions work as follows: For every $\alpha \in S_M$, $\theta_\alpha(v)$ is the composition vector that is obtained by rearranging the entries of v in the specific way that α permutes the elements of $\mathcal{L}(M)$. Note that, for every $v \in V(M,N)$ and every $\pi \in v$, we have $\alpha(\pi) \in \theta_\alpha(v)$. It is easy to check that these functions satisfy the conditions mentioned in definition 2.6. Through proposition 2.1, this group action partitions $V(M,N)$ into orbits $\phi_1, \phi_2, ..., \phi_L$ each of which is a set of AECs. Note that the family of sets $\{r_l\}_{1 \le l \le L}$, defined by $r_l = \bigcup_{v \in \phi_l} v$, forms a partition of the set of all preference profiles. We call each of these sets an *anonymous and neutral equivalence class* (ANEC). That is, each set is the class of all preference profiles that are equivalent such that each preference profile in a set can be generated from another in the same set via permuting the names of voters and/or alternatives. For a given M and N, the set of all ANECs is denoted by $R(M,N)$. It should also be noted that, for any $r_1, r_2 \in R(M,N)$ such that $r_1 \neq r_2$, $v_1 \subseteq r_1$ and $v_2 \subseteq r_2$, there does not exist $\alpha \in S_M$ such that $\theta_\alpha(v_1) = v_2$. That is, it is impossible to obtain a preference profile π' via permuting the names of voters and/or alternatives in another preference profile π that does not belong to the same ANEC that π' does. In what follows, we give an example to clarify the above definitions.

Example 2.1. *Suppose a set $N = \{1,2,3\}$ of voters are to choose from a set $M = \{a,b\}$ of alternatives. Then, $\mathcal{L}(M)$ consists of two preference orderings: p_1, in which a is ranked above b and p_2, in which b is ranked above a. Each preference profile is a list of p_1 and p_2 decisions for each voter and thus, $\mathcal{L}(M)^N = \{\pi_1 = (p_1,p_1,p_1)$, $\pi_2 = (p_1,p_1,p_2)$, $\pi_3 = (p_1,p_2,p_1)$, $\pi_4 = (p_1,p_2,p_2)$, $\pi_5 = (p_2,p_1,p_1)$, $\pi_6 = (p_2,p_1,p_2)$, $\pi_7 = (p_2,p_2,p_1)$, $\pi_8 = (p_2,p_2,p_2)\}$. There are six ways to permute the names of three voters: $\eta_1 = (1)(2)(3)$, $\eta_2 = (12)(3)$, $\eta_3 = (13)(2)$, $\eta_4 = (1)(23)$, $\eta_5 = (123)$, $\eta_6 = (132)$. $S_N = \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\}$ acts on $\mathcal{L}(M)^N$ via the family of functions $\theta_{\eta_k}(\pi) = \pi^{\eta_k}$, $k = 1,2,...,6$. Then, $\mathcal{L}(M)^N$ is partitioned into four AECs such that $O_1 = \{\pi_1\}$, $O_2 = \{\pi_2, \pi_3, \pi_5\}$, $O_3 = \{\pi_4, \pi_6, \pi_7\}$ and $O_4 = \{\pi_8\}$. These orbits are represented by the following composition vectors respectively: $V(M,N) = \{v_1 = (3,0)$, $v_2 = (2,1)$, $v_3 = (1,2)$, $v_4 = (0,3)\}$. There are only two permutations of alternatives: $\alpha_1 = (a)(b)$ and $\alpha_2 = (ab)$. $S_M = \{\alpha_1, \alpha_2\}$ acts on $V(M,N)$ via the family of functions defined as follows: As $\alpha_1(p_k) = p_k$ for $k = 1,2$, θ_{α_1} is the identity function defined on $V(M,N)$. On the other hand, $\alpha_2(p_1) = p_2$ and $\alpha_2(p_2) = p_1$ and hence, θ_{α_2} is the function defined on $V(M,N)$ that maps v_1 to v_4, v_2 to v_3,*

v_3 to v_2, and v_4 to v_1. Group action of S_M partitions $V(M, N)$ into two orbits: $\phi_1 = \{v_1, v_4\}$ and $\phi_2 = \{v_2, v_3\}$. Finally, these two orbits correspond to the following ANECs respectively: $r_1 = \{\pi_1, \pi_8\}$ and $r_2 = \{\pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7\}$.

Definition 2.8. *For every $v \in V(M, N)$, the **stabilizer** of v is the set $(S_M)_v \subseteq S_M$, given by $(S_M)_v = \{\alpha \in S_M | \theta_\alpha(v) = v\}$. An $\alpha \in S_M$ such that $\alpha \in (S_M)_v$ for some $v \in V(M, N)$ is said to **stabilize** v.*

The following remark presents three important statements that we obtain by combining definition 2.8 with the definition of θ_α.

Remark 2.1. $\forall v, w \in V(M, N), \forall \pi \in v, \forall \alpha \in S_M$,

i) $I_{S_M} \in (S_M)_v$ *where* I_{S_M} *is the identity permutation in* S_M*;*

ii) $\alpha(\pi) \in w \Leftrightarrow \theta_\alpha(v) = w$*;*

iii) $\forall p \in \mathcal{L}(M), v(p) = v(\alpha(p)) \Leftrightarrow \alpha \in (S_M)_v$.

Definition 2.9. *For a finite group $(G, \bullet)$ and $g \in G$, the smallest $k \in \mathbb{N}$ such that $g^k = I_G$ is called the **order** of g and is denoted by $o(g)$.*

Proposition 2.2. *Let X be an arbitrary finite set. Then, $\forall \alpha \in S_X$,*

i) If α is an r-cycle, then $o(\alpha) = r$;

ii) If $\alpha = \beta_1\beta_2...\beta_K$ is a product of disjoint r_k-cycles β_k, then $o(\alpha) = lcm\{r_1, r_2, ..., r_K\}$.

Once more, we present this very standard group theoretical proposition without proof. In the following remark, we provide three additional points all of which follow from proposition 2.2.

Remark 2.2. $\forall \alpha \in S_M, \forall k \in \mathbb{N}$ *such that* $1 \leq k \leq m$*,*

i) $o(\alpha) = 1 \Leftrightarrow \alpha = I_M$*;*

ii) $o(\alpha) \mid m!$*;*

iii) $\exists \lambda \in S_M$ *such that* $o(\lambda) = k$.

3 Results

3.1 Existence of Anonymous and Neutral SWFs

Lemma 3.1. *Let* $\alpha \in S_M$. *Then,* $\exists v \in V(M,N)$ *such that* $\alpha \in (S_M)_v \Leftrightarrow o(\alpha) \mid n$.

Proof. $(\Longrightarrow)$: Let $\alpha \in S_M$ and suppose there exists $v \in V(M,N)$ such that $\alpha \in (S_M)_v$. According to remark 2.1, $\forall p \in \mathcal{L}(M), v(p) = v(\alpha(p))$. This implies that $\forall p \in \mathcal{L}(M), \forall k \in \mathbb{N}, v(p) = v(\alpha^k(p))$. Note that, $\forall k, l \in \mathbb{N}$ such that $1 \leq k < l \leq o(\alpha)$, we have $\alpha^k \neq \alpha^l$ and thus $\forall p \in \mathcal{L}(M), \alpha^k(p) \neq \alpha^l(p)$. Hence, $\forall p \in \mathcal{L}(M), o(\alpha) \mid |v^{-1}(v(p))|$. But then, $o(\alpha) \mid \sum_{p \in \mathcal{L}(M)} v(p) = n$.

$(\Longleftarrow)$: Let $\alpha \in S_M$ be such that $o(\alpha) \mid n$. Pick an arbitrary $p \in \mathcal{L}(M)$ and define $\Theta(p) = \{q \in \mathcal{L}(M) \mid \exists k \in \mathbb{N}$ such that $q = \alpha^k(p)\}$. Note that $|\Theta(p)| = o(\alpha)$. Now consider the function $v : \mathcal{L}(M) \to \mathbb{N}$ defined by:

$$v(q) = \begin{cases} n/o(\alpha) & \text{if } q \in \Theta(p) \\ 0 & \text{if } q \notin \Theta(p) \end{cases}.$$

As $\sum_{q \in \mathcal{L}(M)} v(q) = \sum_{q \in \Theta(p)} n/o(\alpha) = n$, v is an AEC. Furthermore, as

$$[\theta_\alpha(v)](q) = v(\alpha(q)) = \begin{cases} n/o(\alpha) & \text{if } \alpha(q) \in \Theta(p) \\ 0 & \text{if } \alpha(q) \notin \Theta(p) \end{cases} = \begin{cases} n/o(\alpha) & \text{if } q \in \Theta(p) \\ 0 & \text{if } q \notin \Theta(p) \end{cases} = v(q),$$

we have $\alpha \in (S_M)_v$. □

Proposition 3.1. $\exists \alpha \in S_M$ *such that* $1 \neq o(\alpha) \mid n \Leftrightarrow gcd(m!, n) \neq 1$.

Proof. $gcd(m!, n) \neq 1 \Leftrightarrow \exists d \in \{1, 2, ..., m\}$ such that $d \mid n$. Then, remark 2.2 completes the proof. □

Lemma 3.2. $\forall r \in R(M,N), \forall v, w \subseteq r, \exists! \alpha \in S_M$ *such that* $\theta_\alpha(v) = w \Leftrightarrow gcd(m!, n) = 1$.

Proof. Pick an arbitrary $r \in R(M,N)$ and $v, w \subseteq r$. Note that, by definition, $\exists \alpha \in S_M$ such that $\theta_\alpha(v) = w$ regardless of the relation between m and n. Now, let us deal with uniqueness.

$(\Longrightarrow)$: For a proof by contrapositive, suppose that $gcd(m!, n) \neq 1$. Then, according to lemma 3.1 and proposition 3.1, $\exists v \in V(M,N), \exists \lambda \in (S_M)_v$ such that $\lambda \neq I_M$. Now, pick $\alpha \in S_M$ such that $\theta_\alpha(v) = w$. Then, $\theta_{\alpha \circ \lambda}(v) = \theta_\alpha(\theta_\lambda(v)) = \theta_\alpha(v) = w$. As $\lambda \neq I_M$, we have $\alpha \circ \lambda \neq \alpha$ and we're done.

($\Longleftarrow$) : Suppose $gcd(m!, n) = 1$ and there exists $\alpha, \lambda \in S_M$ such that $\theta_\alpha(v) = \theta_\lambda(v) = w$. Since $\theta_\alpha \circ \theta_{\alpha^{-1}} = \theta_{\alpha \circ \alpha^{-1}} = I$, where I is the identity function on $V(M, N)$, we have $\theta_{\alpha^{-1} \circ \lambda}(v) = \theta_{\alpha^{-1}}(\theta_\lambda(v)) = \theta_{\alpha^{-1}}(w) = v$. Thus, $\alpha^{-1} \circ \lambda \in (S_M)_v$ and according to lemma 3.1 and proposition 3.1, $\alpha^{-1} \circ \lambda = I_M$. Hence, $\alpha = \lambda$. □

Now, we present the main result of this subsection. Theorem 3.1 is a direct result of lemmas 3.1 and 3.2, together with proposition 3.1.

Theorem 3.1. $\exists f : \mathcal{L}(M)^N \to \mathcal{L}(M)$ *that is anonymous and neutral* $\Leftrightarrow$ *gcd*$(m!, n) = 1$.

Proof. ($\Longrightarrow$) : Suppose $gcd(m!, n) \neq 1$ and there exists $f : \mathcal{L}(M)^N \to \mathcal{L}(M)$ that is anonymous and neutral. According to lemma 3.2, $\exists r \in R(M, N), \exists v, w \subseteq r$ and $\exists \alpha \neq \lambda \in S_M$ such that $\theta_\alpha(v) = \theta_\lambda(v) = w$. Then, $\forall \pi \in v, \forall \pi' \in w, \exists \eta, \mu \in S_N$ such that $\pi' = \alpha(p)^\eta = \lambda(p)^\mu$. Hence, $f(\pi') = \alpha(f(\pi)) = \lambda(f(\pi))$ and this contradicts the fact that $\alpha \neq \lambda$.

($\Longleftarrow$) : Suppose $gcd(m!, n) = 1$ and consider the SWF f constructed as follows:

Pick any $r \in R(M, N)$ and fix some arbitrary AEC $v_r \subseteq r$ and $p_r \in \mathcal{L}(M)$. For all $\pi \in v_r$, define $f(\pi) = p_r$. Furthermore, for any $v \subseteq r$ and $\pi' \in v$, define $f(\pi') = \alpha(p_r)$ where $\alpha \in S_M$ is the unique permutation of alternatives such that $\theta_\alpha(v_r) = v$. Repeat this procedure for all $r \in R(M, N)$.

Clearly, f is a well-defined SWF. To show that it is anonymous and neutral, let us pick some arbitrary $\pi \in v \subseteq r \subseteq \mathcal{L}(M)^N$, $\alpha \in S_M$ and $\eta \in S_N$. Then, $\forall \pi' \in \mathcal{L}(M)^N, \pi' = \alpha(\pi)^\eta \in w$ implies $\theta_\alpha(v) = w$. Suppose, $\lambda, \gamma \in S_M$ are the unique permutations such that $\theta_\lambda(v_r) = v$ and $\theta_\gamma(v_r) = w$. Then, we have $\theta_{\gamma \circ \lambda^{-1}}(v) = \theta_\gamma(\theta_{\lambda^{-1}}(v)) = w$ and by lemma 3.2, $\alpha = \gamma \circ \lambda^{-1}$. Furthermore, due to the way f is constructed, we have $f(\pi') = \gamma(p_r)$ and $f(\pi) = \lambda(p_r)$. Hence, $f(\alpha(\pi)^\eta) = f(\pi') = \gamma(p_r) = \alpha(\lambda(p_r)) = \alpha(f(\pi))$. □

Theorem 3.1 provides a necessary and sufficient condition, $C_1 : gcd(m!, n) = 1$, for the existence of an anonymous and neutral SWF. An immediate question at this point is how likely C_1 is. We devote the rest of this subsection to computing the probability that C_1 is satisfied assuming that m is fixed and n varies over $\mathbb{N}$ with each natural number being equally likely. As such, this probability corresponds to the asymptotic density of the set $\Gamma = \{n \in \mathbb{N} \mid C_1\}$. Hence, $\Pr(C_1 \mid m) = \lim_{n\to\infty} \frac{|\Gamma_n|}{n}$ where $\Gamma_n = \{k \in \Gamma \mid k \leq n\}$, if such a limit exists.

Proposition 3.2. *Suppose* $p_T \leq m < p_{T+1}$ *where* p_t *denotes the* t^{th} *prime number. Then,* $\Pr(C_1 \mid m) = \prod_{t=1}^{T}(1 - \frac{1}{p_t})$.

Proof. $p_T \leq m < p_{T+1}$ implies that $\Gamma_n = \{k \in \mathbb{N} \mid k \leq n$ and $gcd(p_t, k) = 1$ for all $t \in \{1, 2, ..., T\}\}$. Let $\Gamma_n^t = \{k \in \mathbb{N} \mid k \leq n$ and $gcd(p_t, k) \neq 1\}$. Using De Morgan's Law together with the inclusion-exclusion principle, we have:

$$|\Gamma_n| = n - \sum_{\varnothing \neq S \subseteq \{1,2,\ldots,T\}} (-1)^{|S|+1} \left| \bigcap_{t \in S} \Gamma_n^t \right|.$$

Moreover, $\left| \bigcap_{t \in S} \Gamma_n^t \right| = \lfloor \frac{n}{\prod_{t \in S} p_t} \rfloor$ since this is the number of naturals that are less than or equal to n and are multiples of all p_t such that $t \in S$. Substituting in $|\Gamma_n|$, dropping the rounding operator as $n \to \infty$ and canceling n, we get:

$$\lim_{n \to \infty} \frac{|\Gamma_n|}{n} = 1 - \sum_{S \subseteq \{1,2,\ldots,T\}} (-1)^{|S|+1} . \frac{1}{\prod_{t \in S} p_t} = \prod_{t=1}^{T} (1 - \frac{1}{p_t}).$$

□

The following table gives the value of $\Pr(C_1 \mid m)$ for some selected values of m.

TABLE 1

m	$\Pr(C_1 \mid m)$
2	$(1 - \frac{1}{2}) = \frac{1}{2}$
3	$(1 - \frac{1}{2})(1 - \frac{1}{3}) = \frac{1}{3}$
4	$(1 - \frac{1}{2})(1 - \frac{1}{3}) = \frac{1}{3}$
5	$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = \frac{4}{15}$
7	$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7}) = \frac{8}{35}$
11	$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11}) = \frac{16}{77}$
23	$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})...(1 - \frac{1}{23}) \cong 0.164$
37	$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})...(1 - \frac{1}{37}) \cong 0.148$

Evidently, as m increases, the likelihood that n is relatively prime with $m!$ weakly decreases. Furthermore, $\lim_{m \to \infty} \Pr(C_1 \mid m) = 0$, since Euler's product formula yields $\prod_{p \text{ prime}} (1 - \frac{1}{p_t})^{-1} = \sum_{k=1}^{\infty} \frac{1}{k}$ where the expression in the left hand side is the reciprocal of $\lim_{m \to \infty} \Pr(C_1 \mid m)$ and the right hand side is the well-known harmonic series which is divergent.

3.2 Existence of Anonymous and Neutral SCFs

In this subsection, we move on to the domain of SCFs and present an analogous existence result for that domain. This time, the incompatibility between anonymity and neutrality is not strong enough to imply that C_1 is a necessary condition for the existence of anonymous and neutral SCFs. The following example by Moulin (1983) is given to illustrate this point.

Example 3.1. *Suppose a set $N = \{1, 2\}$ of voters are choosing from a set $M = \{a, b, c\}$ of alternatives. Note that, $m = 3$, $n = 2$ and $gcd(m!, n) = 2$. Consider the SCF defined as follows: If the voters agree on their individual best, pick that alternative as the social outcome. Otherwise, pick the remaining (and possibly pareto inferior) alternative. This SCF is clearly anonymous and neutral although C_1 is not met.*

The proof of theorem 3.1 reveals the reason why the set of conditions that work in the context of SWFs are not enough to yield an impossibility result in the context of SCFs. In the former case, obtaining $f(\pi) = \alpha(f(\pi))$ for some $\pi \in \mathcal{L}(M)^N$ and $\alpha \neq I_M$ is enough to reach a contradiction as $f(\pi)$ is a linear order on M. However, in the latter case, procuring $F(\pi) = \alpha(F(\pi))$ for some $\pi \in \mathcal{L}(M)^N$ and $\alpha \neq I_M$ is not enough to seal the proof, as $F(\pi)$ is just one of the alternatives and it is possible that α fixes it. We present two different approaches to overcome this complication and acquire necessary and sufficient conditions for the existence of anonymous and neutral SCFs. Both of these approaches are initially introduced by Moulin (1988a). Here we provide group-theoretical versions of the proofs in order to study the existence of both anonymous and neutral SWFs and SCRs in a single framework.

Since C_1 is no longer a necessary condition for the existence of anonymous and neutral SCFs, we weaken it minimally in order to obtain another condition which is necessary yet remains sufficient. This weaker condition prevents the existence of a permutation of alternatives without a fixed point which is a stabilizer for some AEC. Proposition 3.3 is the analogue of proposition 3.1 in this context and the proof of theorem 3.2 is presented along the same lines with that of theorem 3.1.

Proposition 3.3. *$\exists \alpha \in S_M$ that satisfies $1 \neq o(\alpha) \mid n$ and has no fixed points $\Leftrightarrow$ m can be written as the sum of nontrivial divisors of n, i.e., $\exists z_1, z_2, ..., z_K \in \mathbb{N}$ such that $\forall k \in \{1, 2, ..., K\}, z_k \mid n$ and $z_k \geq 2$, and $\sum\limits_{k=1}^{K} z_k = m$.*

Proof. $(\Longrightarrow)$: Suppose, for a given collection of disjoint z_k-cycles β_k, $\alpha = \beta_1\beta_2...\beta_K \in S_M$ satisfies $1 \neq o(\alpha) \mid n$ and has no fixed points. According to Proposition 2.2, this implies that $\forall k \in \{1, 2, ..., K\}, z_k \geq 2, lcm\{z_1, z_2, ..., z_K\} \mid n$ and $\sum\limits_{k=1}^{n} z_k = m$. Since $lcm\{z_1, z_2, ..., z_K\} \mid n$ implies that $\forall k \in \{1, 2, ..., K\}, z_k \mid n$, we have the desired result.

$(\Longleftarrow)$: Suppose $\exists z_1, z_2, ..., z_K \in \mathbb{N}$ such that $\forall k \in \{1, 2, ..., K\}, z_k \mid n$ and $z_k \geq 2$, and $\sum\limits_{k=1}^{n} z_k = m$. Fix a partition $\beta_1, \beta_2, ..., \beta_K$ of M such that $|\beta_k| = z_k$ and consider $\alpha = \widetilde{\beta_1}\widetilde{\beta_2}...\widetilde{\beta_K} \in S_M$ where $\forall k \in \{1, 2, ..., K\}$, $\widetilde{\beta_k}$ is any ordering of the elements in β_k. Then, $1 \neq o(\alpha) \mid n$ and α has no fixed points. $\square$

Lemma 3.3. *$\forall v \in V(M,N), \exists a_v \in M$ s.t. $\alpha \in (S_M)_v$ implies $\alpha(a_v) = a_v$ if m cannot be written as the sum of nontrivial divisors of n.*

Proof. Suppose $v \in V(M,N)$ is given and m cannot be written as the sum of nontrivial divisors of n. We obtain the alternative a_v using the following algorithm:

Pick an arbitrary $\pi \in v$ and $\forall n \in \mathbb{N}_+$, let $D_n = \{x \in \mathbb{N} \mid x$ can be written as the sum of nontrivial divisors of $n\}$ with the convention that $0 \in D_n$.

Step1: For all $0 \leq t \leq n$, let X_t denote the set of alternatives in M that are ranked first by exactly t voters at profile π. Since $\sum_{0 \leq t \leq n} |X_t| = m$ and $m \notin D_n, \exists t^* \in \{0, 1, ..., n\}$ such that $|X_{t^*}| \notin D_n$. Furthermore, $|X_{t^*}| < m$ since otherwise we would have $mt^* = n$, which would contradict $m \notin D_n$. Hence, $X_{t^*} \subset M$. Pick one such t^* and relabel $X_{t^*} = M_1^\pi$. Continue to step 2 if $|M_1^\pi| > 1$, otherwise terminate the algorithm.

Step2: For all $0 \leq t \leq n$, let X_t denote the set of alternatives in M_1^π that are ranked first by exactly t voters in the restricted profile $\pi_1 = \pi_{|M_1^\pi}$. Similarly, $\sum_{0 \leq t \leq n} |X_t| = |M_1^\pi|$ and $|M_1^\pi| \notin D_n \Rightarrow \exists t^* \in \{0, 1, ..., n\}$ such that $|X_{t^*}| \notin D_n$ with $X_{t^*} \subset M_1^\pi$. Pick one such t^* and relabel $X_{t^*} = M_2^\pi$. Continue to the next step if $|M_2^\pi| > 1$, otherwise terminate the algorithm.

Step3: For all $0 \leq t \leq n$, let X_t denote the set of alternatives in M_2^π that are ranked first by exactly t voters in the restricted profile $\pi_2 = \pi_{1|M_2^\pi}$. Once again, $\sum_{0 \leq t \leq n} |X_t| = |M_2^\pi|$ and $|M_2^\pi| \notin D_n \Rightarrow \exists t^* \in \{0, 1, ..., n\}$ such that $|X_{t^*}| \notin D_n$ with $X_{t^*} \subset M_2^\pi$. Pick one such t^* and relabel $X_{t^*} = M_3^\pi$. Continue to the next step if $|M_3^\pi| > 1$, otherwise terminate the algorithm.

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This algorithm terminates after a finite number of steps since at each step we dismiss a strictly positive number of alternatives and M is finite. Suppose it terminates at step T. Then, $|M_T^\pi| > 0$ as $|M_T^\pi| \notin D_n$. Furthermore, $|M_T^\pi| < 2$, since otherwise the algorithm would not have terminated. Hence, $|M_T^\pi| = 1$. Let $\{a_v\} = M_T^\pi$.

To prove that a_v is a fixed point of all permutations of alternatives that stabilize v, pick an arbitrary $\alpha \in (S_M)_v$. Then, $\exists \eta \in S_N$ such that $\alpha(\pi)^\eta = \pi$. Since the algorithm is itself anonymous and neutral, we can pick $(M_t^{\alpha(\pi)^\eta})_{t=1,2,...,T}$ such that $\forall t \in \{1, 2, ..., T\}, M_t^\pi = M_t^{\alpha(\pi)^\eta} = M_t^{\alpha(\pi)} = \alpha(M_t^\pi)$. In particular, $M_T^\pi = \alpha(M_T^\pi)$ and thus, $\alpha(a_v) = a_v$. □

Lemma 3.3 demonstrates that, for any $v \in V(M,N)$, the set of alternatives $M_v = \{a \in M \mid \forall \alpha \in (S_M)_v, \alpha(a) = a\}$ is nonempty, given that m cannot be written as the sum of nontrivial divisors of n. In the following theorem, we use these sets to construct an anonymous and neutral SCF.

Theorem 3.2. *$\exists F : \mathcal{L}(M)^N \to M$ that is anonymous and neutral $\Leftrightarrow$ m cannot be written as the sum of nontrivial divisors of n.*

Proof. $(\Longrightarrow)$: Suppose m can be written as the sum of nontrivial divisors of n and there exists $F : \mathcal{L}(M)^N \to M$ that is anonymous and neutral. Lemma 3.1 and proposition 3.3 together imply that $\exists v \in V(M,N), \exists \alpha \in (S_M)_v$ such that α has no fixed points. Now, pick a preference profile $\pi \in v$. Once again, $\alpha(\pi) \in v$ and $\exists \eta \in S_N$ such that $\pi^\eta = \alpha(\pi)$. Then, by anonymity and neutrality of f, we get $F(\pi) = F(\pi^\eta) = F(\alpha(\pi)) = \alpha(F(\pi))$. This contradicts with α not having fixed points.

$(\Longleftarrow)$: Suppose m cannot be written as the sum of nontrivial divisors of n and consider the SCF constructed as follows:

Pick any $r \in R(M,N)$ and fix some arbitrary AEC $v_r \subseteq r$ and $a_r \in M_v$. For all $\pi \in v_r$, define $F(\pi) = a_r$. Furthermore, for any $\pi' \in v \subseteq r$, define $F(\pi') = \alpha(a_r)$ where $\alpha \in S_M$ satisfies $v = \theta_\alpha(v_r)$. Repeat this procedure for all $r \in R(M,N)$.

To prove that F is well-defined, we have to show that $\forall r \in R(M,N), \forall v, w \subseteq r, \forall x \in M_v$ and $\forall \alpha, \lambda \in S_M$ such that $w = \theta_\alpha(v) = \theta_\lambda(v)$, we have $\alpha(a) = \lambda(a)$. This follows from the fact that $\alpha \circ \lambda^{-1} \in (S_M)_v$ and hence $(\alpha \circ \lambda^{-1})(a) = a$. Moreover, to show that F is anonymous and neutral, let us pick some $\pi \in v \subseteq r$, $\alpha \in S_M$ and $\eta \in S_N$. Suppose, $\pi' = \alpha(\pi)^\eta \in w \subseteq r$ and let $\lambda \in S_M$ satisfy $\theta_\lambda(v_r) = v$. Then, $\theta_{\alpha\circ\lambda}(v_r) = w$ and we have $F(\pi) = \lambda(a_r)$ and $F(\pi') = (\alpha \circ \lambda)(a_r)$. This implies that $F(\alpha(\pi)^\eta) = F(\pi') = (\alpha \circ \lambda)(a_r) = \alpha(F(\pi))$. □

It is fairly evident that the divisibility condition required in theorem 3.2 (C_2: m cannot be written as the sum of nontrivial divisors of n) is implied by C_1. To be able to assess how restrictive C_2 is, consider the condition $C_3 : gcd(m,n) = 1$. It is also evident that C_2 implies C_3 and the following corollary is immediate.

Corollary 3.1. $gcd(m,n) \neq 1 \Rightarrow \nexists F : \mathcal{L}(M)^N \to M$ *that is anonymous and neutral.*

Unlike in the case of C_1, there is no closed expression about the likelihood of C_2 to hold. Ever so, $C_1 \Rightarrow C_2 \Rightarrow C_3$ means that we have $\Pr(C_3 \mid m) \geq \Pr(C_2 \mid m) \geq \Pr(C_1 \mid m)$, whenever these limits exist. Since $\lim_{m\to\infty} \Pr(C_3 \mid m)$ does not exist, we do not have a limit value for $\Pr(C_2 \mid m)$. Nevertheless, we present the likelihood that C_2 holds for small values of m in the following table. Clearly, $\Pr(C_2 \mid m)$ is not monotonic in m.

TABLE 2

m	$\Pr(C_2 \mid m)$
2	$\frac{1}{2}$
3	$\frac{2}{3}$
4	$\frac{1}{2}$
5	$\frac{2}{3}$
6	$\frac{1}{3}$
7	$\frac{9}{14}$
8	$\frac{7}{15}$

As a second approach to overcome the impediment regarding C_1 not being a necessary condition for the existence of anonymous and neutral SCFs, we strengthen the requirements to be fulfilled by a SCF and add paretian optimality in addition to anonymity and neutrality.

Definition 3.1. *A SCF is* ***paretian (or paretian optimal)*** *if* $\forall a, b \in M$ *and* $\forall \pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N$, ap^ib *for all* $i \in N \Rightarrow b \notin F(\pi)$.

Definition 3.2. *For the following theorem, we pick a specific* $\alpha \in (S_M)$ *and construct an appropriate preference profile in which none of the pareto optimal alternatives is a fixed point of* α. *This way, we establish* C_1 *to be the necessary and sufficient condition for the existence of an anonymous, neutral and paretian SCF.*

Theorem 3.3. $\exists F : \mathcal{L}(M)^N \to M$ *that is anonymous, neutral and paretian* $\Leftrightarrow$ $gcd(m!, n) = 1$.

Proof. $(\Longrightarrow)$: We will follow a path similar to the one we followed in the proof of theorem 3.1. Suppose $gcd(m!, n) \neq 1$ and there exists an anonymous, neutral and paretian SCF $F : \mathcal{L}(M)^N \to M$. As before, $gcd(m!, n) \neq 1 \Rightarrow \exists d \in \mathbb{N}$ such that $1 < d \leq m$ and $d \mid n$. This time, we restrict our attention to a specific permutation $\alpha \in S_M$ such that $o(\alpha) = d$ and construct a specific preference profile π to help us get a contradiction. Consider $\alpha \in S_M$ given by the cycle decomposition $\alpha = (a_1a_2...a_d)(a_{d+1})(a_{d+2})...(a_m)$ for some enumeration $a_1, a_2, ..., a_m$ of alternatives and the preference profile $\pi = (\alpha^i(p^*))_{i \in N} \in \mathcal{L}(M)^N$ where $p^* \in \mathcal{L}(M)$ is given by: $a_1p^*a_2p^*...p^*a_m$. First thing to notice is that, as $o(\alpha) = d$ by proposition 2.2, we have $\forall j \in \{1, 2, ..., n-d\}, \alpha^{j+d}(p^*) = \alpha^d(\alpha^j(p^*)) = \alpha^j(p^*)$. Then, for every $j \leq d$, preference profile π contains n/d voters whose preferences are given by $\alpha^j(p^*)$. Furthermore, $\alpha(\pi)$ also contains n/d voters whose preferences are given by $\alpha^j(p^*)$. This implies that there exists $\eta \in S_N$ such that $\alpha(\pi) = \pi^\eta$. By anonymity and neutrality of F, we get $F(\pi) = F(\pi^\eta) = F(\alpha(\pi)) = \alpha(F(\pi))$. As $\alpha(a_j) \neq a_j$ for all $j \leq d$, $F(\pi) \in \{a_{d+1}, a_{d+2}, ..., a_m\}$. But, since $\forall j \in \{1, 2, ..., n\}$ and $\forall k > d, a_1\alpha^j(p^*)a_k$, this contradicts with paretian optimality of F.

$(\Longleftarrow)$: Suppose $gcd(m!, n) = 1$ consider the SCF F constructed as follows: Pick any $r \in R(M, N)$ and fix some arbitrary $v_r \subseteq r$ and $a_r \in M$ such that a_r is preferred to any other alternative in some $\pi \in v_r$ by at least one voter. For any $\pi \in v_r$, define $F(\pi) = a_r$. Furthermore, for any $\pi' \in v \subseteq r$, define $F(\pi') = \alpha(a_r)$ where $\alpha \in S_M$ is the unique permutation of alternatives such that $\theta_\alpha(v_r) = v$. Repeat this procedure for all $r \in R(M, N)$.

Clearly, F is a well-defined SCF. By the same argument from the proof of theorem 3.1, it is anonymous and neutral. Finally, it is paretian since for all $\pi \in \mathcal{L}(M)^N$, there exists an agent who ranks $F(\pi)$ as the best in her preference ordering. $\square$

Theorems 3.1 - 3.3 show that, for certain numbers of voters and alternatives, some compositions of individual preferences possess a cyclical nature that leads to unavoidable ties in the social preference if we impose that the mechanism via which the social preference is determined does not favor any of the voters and/or alternatives. This impedes the existence of a resolute SWR (resp. SCR) that returns a strict social preference ordering (resp. unique outcome) for each preference profile.

3.3 Maximal Domains of Existence

Our results on the existence of anonymous and neutral SWFs (SCFs) and our framework through which we obtain these results allow us to describe a maximal domain for anonymous and neutral SWFs (SCFs) for any given M and N. We define the domain of an anonymous and neutral SWF $D^W(M, N) \subseteq \mathcal{L}(M)^N$ and of such SCF $D^C(M, N) \subseteq \mathcal{L}(M)^N$ based on ANECs since each ANEC contains all preference profiles that have the same "preference structure", i.e., that can be obtained from each other via permuting the names of voters and/or alternatives. Hence,

$$D^W(M, N) = \bigcup_{r \in R^W(M,N)} r \quad \text{and } D^C(M, N) = \bigcup_{r \in R^C(M,N)} r$$

Based on theorem 3.1, it is clear that the necessary and sufficient condition for the existence of anonymous and neutral SWFs is that there exists no AEC which is stabilized by some permutation of alternatives other than the identity permutation. Thus, we define

$$D^W(M, N) = \bigcup_{v \in V^W(M,N)} v \quad \text{where } V^W(M, N) = \{v \in V(M, N) \mid (S_M)_v = \{I_M\}\}.$$

Note that, $\forall r \in R(M, N)$ and $\forall v, w \in V(M, N)$ such that $v \subseteq r$ and $w \subseteq r$, $(S_M)_v = \{I_M\} \Leftrightarrow (S_M)_w = \{I_M\}$.

$D^C(M, N)$ can be defined in a similar fashion. We know that the necessary and sufficient condition for the existence of anonymous and neutral SCFs is that there exists no AEC that is stabilized by some permutation of alternatives which does not have any fixed points. Hence, we define

$$D^C(M, N) = \bigcup_{v \in V^C(R,N)} v \quad \text{where } V^C(M, N) = \{v \in V(M, N) \mid M_v \neq \varnothing\}.$$

As before we have, $\forall r \in R(M,N)$ and $\forall v, w \in V(M,N)$ such that $v \subseteq r$ and $w \subseteq r$, $M_v \neq \varnothing \Leftrightarrow M_w \neq \varnothing$.

The following points can easily be noted on $D^W(M,N)$ and $D^C(M,N)$. First of all, since C_1 implies C_2, $D^W(M,N) \subseteq D^C(M,N)$ holds for all M and N. Secondly, theorems 3.1 and 3.2 imply that C_1 holds if and only if $D^W(M,N) = \mathcal{L}(M)^N$ and C_2 holds if and only if $D^C(M,N) = \mathcal{L}(M)^N$. Finally, unlike many maximal domains, $D^W(M,N)$ and $D^C(M,N)$ are, in fact, the maximum domains which contain entire ANECs, and over which an anonymous and neutral SWF (SCF) can be defined.

4 Anonymous and Neutral Refinement of SCRs

4.1 Anonymous and Neutral Tie-Breaking Procedure

Resoluteness is one of the major requirements for social decision making mechanisms since in many contexts individual preferences are to be aggregated into a single social decision. Among numerous examples, one can consider the presidential elections, referendums, legislative voting, etc. However, there is no anonymous, neutral and paretian SCR which satisfies resoluteness. Since breaking ties is the most natural way to obtain a resolute refinement of a SCR, the procedure through which ties are to be broken stands as a major concern. This concern provides the motivation for the analysis of resolute refinements of familiar SCRs.

A SCR $\tilde{F}$ is called a *refinement* of a SCR F if $\forall \pi \in \mathcal{L}(M)^N$, $\tilde{F}(\pi) \subseteq F(\pi)$. A *resolute refinement* of a SCR F is a refinement of F that is resolute. Moulin (1988a, 1988b) proposes a *tie-breaking agenda* that breaks ties using a strict ordering of the alternatives. According to Moulin's approach, a SCF $F^{\succ}$ is obtained from a (possibly) irresolute SCR F using $\succ \in \mathcal{L}(M)$ as the tie-breaking agenda by: For any $\pi \in \mathcal{L}(M)^N$, $F^{\succ}(\pi)$ is the maximum element of $F(\pi)$ with respect to $\succ$. This approach has been adopted in many studies since then.[4] Another familiar approach to obtain a resolute refinement of a SCF is to break the ties according to the relative rankings of the alternatives in tie given the preference ranking of one of the voters. That is, given a *tie-breaking voter* $i \in N$, a SCF F^i can be obtained from a (possibly) irresolute SCR F by: For any $\pi \in \mathcal{L}(M)^N$, $F^i(\pi)$ is the maximum element of $F(\pi)$ with respect to p^i. In other words, tie-breaking ordering is not fixed over all preference profiles, but rather it is determined by voter i's preference at each profile.

Both of the tie-breaking procedures described above are easy to implement to obtain resolute refinements of SCRs. However, the resolute refinements obtained through these procedures do not preserve the anonymity and neutrality properties of the original rule, i.e., $F^{\succ}$ is not neutral and F^i is not anonymous.

[4]For a general comparison on how different methods of handling ties (including the tie-breaking agenda) fare in terms of monotonicity properties see Sanver and Zwicker (2010).

In what follows, we propose an alternative way for breaking the ties that prevail among the winners without undermining the anonymity and the neutrality of a SCR. Our proposal is using the outcome of an anonymous, neutral and neutral SWR to break the ties. Since C_1 is the necessary and sufficient condition to ensure such rules exist, the procedure that we define is valid if and only if C_1 holds.

Definition 4.1. *For any given SCR F and resolute SWR f, the resolute refinement of F with respect to f is defined by:* $\forall \pi \in \mathcal{L}(M)^N$, $F^f(\pi) = \underset{a \in F(\pi)}{\arg\max}\, f(p)$

Definition 4.2. *For any given anonymous and neutral SCR F, an anonymous and neutral resolute (ANR) refinement of F is F^f where f is any anonymous and neutral SWF.*

It can easily be seen that, for any given ANR refinements of anonymous and neutral SCRs are anonymous, neutral and resolute. In the next section, we analyze the conditions under which they are monotonic.

4.2 Resolute Refinements of SCRs and Simple Monotonicity

Simple monotonicity is the most basic of all monotonicity axioms and satisfied by almost all of the familiar SCRs. It asserts that if a voter lifts the rank of an alternative in his preference ranking as he keeps the relative rankings of the remaining alternatives fixed, this should not be detrimental to the lifted alternative's chances of being a winner. Clearly, the resolute refinement of a simple monotonic SCR might not be simple monotonic itself. In this subsection, we compare and contrast the conditions for the resolute refinements of simple monotonic SCRs that are obtained via the conventional tie-breaking procedures ($F^{\succ}$ and F^i) with ANR procedure (F^f) to satisfy the axiom as well.

Definition 4.3. *Given $a \in M$ and $\pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N$, we say that $\pi' = (q^i)_{i \in N} \in \mathcal{L}(M)^N$ is a* ***simple lifting for a with respect to π*** *if*

i) $\exists i \in N$ *such that* $\forall b, c \in M \setminus \{a\}, ap^ib \Rightarrow aq^ib$ *and* $bp^ic \Longleftrightarrow bq^ic$

ii) $\forall j \in N \setminus \{i\}$, $p^j = q^j$.

Definition 4.4. *A SCR $F : \mathcal{L}(M)^N \to 2^M \setminus \{\varnothing\}$ is called* ***simple monotonic*** *(SM) if* $\forall i \in N, \forall \pi \in \mathcal{L}(M)^N, \forall a \in F(\pi)$, *$\pi'$ is a simple lifting for a with respect to $\pi \Rightarrow a \in F(\pi')$.*

Let us now introduce a monotonicity axiom which is stronger than simple monotonicity. Exclusive monotonicity requires that simple lifting of an alternative does not lead to a enlarged set of winners. Below, we show that exclusive monotonicity is the necessary and sufficient condition so that $F^{\succ}$ is SM, and it is just the sufficient condition for F^i to be SM.

Definition 4.5. *A SCR $F : \mathcal{L}(M)^N \to 2^M \setminus \{\varnothing\}$ is called **exclusively monotonic** (EM) if $\forall i \in N, \forall \pi \in \mathcal{L}(M)^N, \forall a \in F(\pi)$, π' is a simple lifting for a with respect to $\pi \Rightarrow a \in F(\pi') \subseteq F(\pi)$.*

Proposition 4.1. *For any given neutral SCR F, $\forall \succ \in \mathcal{L}(M), F^{\succ}$ is SM $\Leftrightarrow F$ is EM.*

Proof. Note that, since F is neutral, $\exists \succ \in \mathcal{L}(M)$ such that $F^{\succ}$ is SM $\Rightarrow \forall \succ \in \mathcal{L}(M), F^{\succ}$ is SM.

$(\Longrightarrow)$: Suppose F is not EM. Then, $\exists \pi, \pi' \in \mathcal{L}(M)^N, \exists a \in F(\pi)$ such that π' is a simple lifting for a with respect to π and either $a \notin F(\pi')$ or $F(\pi') \nsubseteq F(\pi)$. If $a \notin F(\pi')$, then for any $\succ \in \mathcal{L}(M)$ such that $\forall b \in M \setminus \{a\}, a \succ b$, we have $a = F^{\succ}(\pi)$ and $a \neq F^{\succ}(\pi')$. If $F(\pi') \nsubseteq F(\pi)$, then for any $\succ \in \mathcal{L}(M)$ such that $b \succ a \succ c$ for all $c \in F(\pi) \setminus \{a\}$ and for some $b \in F(\pi') \setminus F(\pi)$, we have $a = F^{\succ}(\pi)$ and $a \neq F^{\succ}(\pi')$. Hence, in either case, $F^{\succ}$ is not SM.

$(\Longleftarrow)$: Suppose F is EM. Then, $\forall \succ \in \mathcal{L}(M), \forall \pi, \pi' \in \mathcal{L}(M)^N$ such that π' is a simple lifting for $a = F^{\succ}(\pi)$ with respect to π, we have $a = F^{\succ}(\pi')$ since $a = \underset{b \in F(\pi)}{\arg\max} \succ \Rightarrow a = \underset{b \in F(\pi')}{\arg\max} \succ$ when $F(\pi') \subseteq F(\pi)$. □

Note that this statement is not correct when we consider F^i. The following proposition shows that EM is sufficient for F^i to preserve SM. It should also be noted that since F is anonymous, F^i is SM for some $i \in N$ implies F^i is SM for all $i \in N$.

Proposition 4.2. *For any given anonymous SCR F, F is EM $\Rightarrow \forall i \in N, F^i$ is SM.*

Proof. Suppose F is EM. Then, $\forall i \in N, \forall \pi = (p^i)_{i \in N}, \pi' = (q^i)_{i \in N} \in \mathcal{L}(M)^N$ such that π' is a simple lifting for $a = F^i(\pi)$ with respect to π, we have $a = F^i(\pi')$ since $a = \underset{b \in F(\pi)}{\arg\max}\, p^i \Rightarrow a = \underset{b \in F(\pi')}{\arg\max}\, q^i$ when we have $F(\pi') \subseteq F(\pi)$ and $\forall b \in M, ap^ib \Rightarrow aq^ib$. □

Let us now consider the compatibility of F^f with simple monotonicity. Firstly, we observe that, depending on the SCR F in focus, whether the ANR refinement F^f satisfies SM or not may depend on f. Then, we show that EM is sufficient for the existence of an ANR refinement that satisfies SM. We define two well-known SCRs, Borda rule and plurality rule, as we employ them in our proofs below.

Definition 4.6. *The **Borda rule** is the SCR defined as: $\forall \pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N$, $F_B(\pi) = \underset{a \in M}{\arg\max}\, s_B^a(\pi)$ where $s_B^a(\pi) = \sum_{i \in N} |\{b \in M \mid ap^ib\}|$.*

Definition 4.7. *The **plurality rule** is the SCR defined as: $\forall \pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N$, $F_P(\pi) = \underset{a \in M}{\arg\max}\, s_P^a(\pi)$ where $s_P^a(\pi) = |\{i \in N \mid \forall b \in M \setminus \{a\}, ap^ib\}$.*

The lemma below shows that if a SCR is EM, then some of its ANR refinements satisfy simple monotonicity.

Lemma 4.1. *Let $gcd(m!, n) = 1$. Then, for any given anonymous, neutral and EM SCR F, there exists an ANR refinement of F that is SM.*

Proof. Let $gcd(m!, n) = 1$ and pick any anonymous, neutral and EM SCR F. Essentially, we show that breaking ties according to the order given by the Borda scores of alternatives will result in a SM ANR refinement. Pick any anonymous, neutral and resolute SWR f. Define the Borda SWR as follows: $\forall \pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N, \forall a, b \in M, a f_B(\pi) b \Leftrightarrow s_B^a(\pi) > s_B^b(\pi)$ or $[s_B^a(\pi) = s_B^b(\pi)$ and $a f(\pi) b]$. Since Borda scores are anonymous and neutral, f_B is anonymous and neutral. Furthermore, F is also resolute since $a f_B(\pi) b$ and $b f_B(\pi) a \Rightarrow a f(\pi) b$ and $b f(\pi) a$, which contradicts that f is itself resolute. Then, F^{f_B} is an ANR refinement of F. To see that it is also SM, pick any $i \in N$ and $\pi, \pi' \in \mathcal{L}(M)^N$ such that π' is a simple lifting for $a = F^{f_B}(\pi)$ with respect to π. Then, $a \in F(\pi)$ and since F is EM, $a \in F(\pi')$. Moreover, $a \in \underset{b \in F(\pi)}{\arg\max}\, s_B^b(\pi) \Rightarrow \{a\} = \underset{b \in F(\pi')}{\arg\max}\, s_B^b(\pi')$. Hence, $a = F^{f_B}(\pi')$. □

Proposition 4.3. *All resolute refinements of F_B are SM.*

Proposition 4.3 is obtained by the arguments that we use in the proof of lemma 4.1. Hence, we present it without proof. In what follows, we show that exclusive monotonicity is not a sufficient condition for all ANR refinements of a SCR to be simple monotonic. Observe that the plurality rule is EM.

Proposition 4.4. *Let $gcd(m!, n) = 1$. For any $m \geq 3$ and $n \geq 2$, there exists anonymous, neutral and resolute SWR f such that F_P^f is not SM.*

Proof. $gcd(m!, n) = 1, m \geq 3, n \geq 2 \Rightarrow \exists k \geq 2$ such that $n = 2k + 1$. Consider some enumeration of alternatives $a_1, a_2, , ..., a_m$ and the preference profile $\pi = (p^i)_{i \in N} \in \mathcal{L}(M)^N$ where

$$p^i = \begin{cases} a_1 > a_2 > a_3 > ... > a_m & \text{if } i = 1, 2, ..., k \\ a_2 > a_1 > a_3 > ... > a_m & \text{if } i = k+1, k+2, ..., 2k. \\ a_m > a_{m-1} > ... > a_2 > a_1 & \text{if } i = 2k+1 \end{cases}$$

Now, let π' be the profile obtained from π as the last voter changes her vote from p^{2k+1} to $\tilde{p}^{2k+1} = a_m > a_{m-1} > ... > a_1 > a_2$, by lifting a_1 over a_2. Notice that $\pi' = \alpha(\pi)^\eta$ where $\alpha = (a_1 a_2)(a_3)...(a_m)$ and

$$\eta(i) = \begin{cases} i + k & \text{if } i = 1, 2, ..., k \\ i - k & \text{if } i = k+1, k+2, ..., 2k. \\ i & \text{if } i = 2k+1 \end{cases}$$

Note that $F_P(\pi) = F_P(\pi') = \{a_1, a_2\}$. Now, consider any anonymous, neutral and resolute SWR f that satisfies $a_1 f(\pi) a_2$. By our previous arguments, such a SWR clearly exists. Then, $F_P^f(\pi) = a_1$. Since $f(\pi') = \alpha(f(\pi))$, we have $a_2 f(\pi') a_1$ and thus, $F_P^f(\pi') = a_2$. Hence, F_P^f is not SM. $\square$

Let us now define a stronger monotonicity property that guarantees that all ANR refinements are SM.

Definition 4.8. *A SCR F is called* ***sensitively monotonic(SEM)*** *if $\forall i \in N, \forall \pi \in \mathcal{L}(M)^N, \forall a \in F(\pi)$, π' is a simple lifting for a with respect to $\pi \Rightarrow \{a\} = F(\pi')$.*

Clearly, Borda rule is SEM while plurality rule is not. It should be noted that sensitive monotonicity implies exclusive monotonicity which implies simple monotonicity. However, these three monotonicity requirements are equivalent on the domain of resolute SCRs. Below we show that the necessary and sufficient condition for all ANR refinements of a SCR to satisfy simple monotonicity is that the SCR is SEM.

Lemma 4.2. *Let $gcd(m!, n) = 1$. Then, for any given anonymous, neutral SCR F, F^f is SM for any anonymous, neutral and resolute SWR $f \Leftrightarrow F$ is SEM.*

Proof. $(\Longrightarrow)$: Let $gcd(m!, n) = 1$. Pick any anonymous, neutral SCR F and suppose it is not SEM. Hence, $\exists i \in N, \exists \pi, \pi' \in \mathcal{L}(M)^N$ such that π' is a simple lifting for $a \in F(\pi)$ with respect to π and $\{a\} \neq F(\pi')$. Pick any $b \in F(\pi') \setminus \{a\}$. We consider 2 cases:
Case 1: $\exists r, r' \in R(M, N)$ such that $r \neq r', \pi \in r$ and $\pi' \in r'$. Consider any anonymous, neutral and resolute SWR f that satisfies $\forall c \in M \setminus \{a\}, a f(\pi) c$ and $\forall d \in M \setminus \{b\}, b f(\pi') d$. By our previous arguments, such a SWR clearly exists. Then, $F^f(\pi) = a$ and $F^f(\pi') = b$. Hence, F^f is not SM.
Case 2: $\exists r \in R(M, N)$ such that $\pi, \pi' \in r$. Then, $\exists \alpha \in S_M$ and $\exists \eta \in S_N$ such that $\alpha(\pi)^\eta = \pi'$. Moreover, since π' is a simple lifting for $a \in F(\pi)$ with respect to π, $\alpha(a) \neq a$. Then, $\alpha^{-1}(a)\ neqa$ as well. Consider any anonymous, neutral and resolute SWR f that satisfies $\forall c \in M \setminus \{a, \alpha^{-1}(a)\}, a f(\pi) c f(\pi) \alpha^{-1}(a)$. Then, $F^f(\pi) = a$. Furthermore, since $f(\pi') = \alpha(f(p))$, we get $\alpha^{-1}(b) f(\pi) \alpha^{-1}(a) \Rightarrow b f(\pi') a$. Hence, $F^f(\pi') \neq a$ and F^f is not SM.

$(\Longrightarrow)$: Let $gcd(m!, n) = 1$. Pick any anonymous, neutral and SEM SCR F and anonymous, neutral and resolute SWR f. Moreover, pick some $i \in N$ and $\pi, \pi' \in \mathcal{L}(M)^N$ such that π' is a simple lifting for $a = F^f(\pi)$ with respect to π. Then, since F is SEM, $a = F(\pi')$ and thus $a = F^f(\pi')$. $\square$

5 Concluding Remarks

In this paper, we present the existence conditions for ANR SWRs and SCRs in a single framework which is provided through group theory. We show that there exists an ANR SWR if and only if

$gcd(m!, n) = 1$. This condition is shown to be also the necessary and sufficient condition for the existence of ANR and paretian SCRs. We present an algorithm for the construction of such SCRs. We revisit the result by Moulin (1983) that there exists an ANR SCR if and only if m cannot be written as the sum of non-trivial divisors of n. Our proof of this result also provides an algorithm, which can be used to define ANR SCR's, given that the aforementioned condition holds.

When the existence conditions do not hold our group theoretical approach allows us to identify the roots (ANECs) that yield cyclic preference structures which refrain social welfare and social choice rules from producing resolute social decisions. This identification of problematic roots facilitates our definition of the maximal domains for ANR SWRs and for such SCRs. The nature of these maximal domains is generically different than the well-known domains reviewed by Arrow et. al. (2002), and the relationship between them is not immediate. Clearly, this points to a direction for further research.

Eğecioğlu and Giritligil (2013) introduces an algebra package that uniformly generates roots for given m and n. Our identification of problematic roots and definitions of maximal domain for ANR SWRs and SCRs constitute a guide to develop similar algebra packages that generate only "healthy" roots. Such a package can provide a good testbed to explore the likelihood of SWRs or SCRs to fulfill particular criteria through Monte Carlo simulations.

In this paper, we also propose a mechanism to obtain ANR refinements of SCRs and introduce the conditions for these refinements to satisfy simple monotonicity. To investigate the conditions for these refinements to fulfill other properties is surely another immediate direction for further research.

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