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Murat Sertel Center for Advanced Economic Studies

Working Papers

Year 2013 No 1

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A revised version of this draft is to appear in TOP 21:3, 590-611, 2013

manuscript No.
(will be inserted by the editor)

Unanimity and the Anscombe's Paradox

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The date of receipt and acceptance will be inserted by the editor

Abstract We establish a new sufficient condition for avoiding a generalization of the Anscombe's paradox. In a situation where ballots describe positions regarding finitely many yes-or-no issues, the Anscombe's $\alpha-$paradox holds if more than α% of the voters disagree with on a majority of issues with the outcome of issue-wise majority voting. We define the level of unanimity of a set of ballots as the number of issues minus the maximal symmetric diatance between two ballots. We compute for the case of large electorates, the exact level of unanimity above which the Anscombe's $\alpha-$paradox never holds, whatever the distribution of votes among ballots.

Key words Anscombe, Voting Paradox, Majority Rule, Unanimity, Issue-wise voting
JEL Class. D71, D72

1 Introduction

We consider situations of multiple referendum, where an electorate faces finitely many dichotomous issues, or proposals. The Anscombe's paradox (Anscombe, 1976) states that issue-wise majority voting may put a majority of the voters on the losing side on a majority of the issues. Table 1 below illustrates the Anscombe's paradox in the case of 3 alternatives and 5 voters:

TABLE 1

	x_1	x_2	x_3	x_4, x_5
1	1	1	0	0
2	1	0	1	0
3	0	1	1	0

Issues $c = 1, 2, 3$ appears in rows, and columns describes the 5 ballots $x_j = (x_j^1, x_j^2, x_j^3)$, $j = 1, ..., 5$, where $x_j^c = 1$ (resp. 0) means that voter j agrees (resp. disagrees) with proposal c. Voting issue-wise according to the majority rule results in choosing 0 for each issue. Hence, the first 3 voters disagree with the outcome on two thirds of the issues.

The Anscombe's paradox is discussed in Nurmi (1999), and Saari (2001). Furthermore, Wagner (1983) shows that the paradox never holds when at least three-fourths of the voters agree on average with the final decision on each of the issues. Hence, voting situations where proposals are adopted or rejected, on average, by a sufficiently strong consensus cannot face the paradox. Put differently, if, on average, voters are not too far from issue-wise unanimity, then the Anscombe's paradox cannot hold.

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The 'Rule of Three-Fourths'[1] is extended in Wagner (1984) to the 'Rule of $(1-\frac{\alpha}{2})$': given a number J of voters, call coalition on some issue c the subset of agents agreeing with the decision regarding c; the rule states that *when the prevailing coalitions, across all issues, comprise on average* $(1-\frac{\alpha}{2})J$ *voters, the set of voters who disagree with more than half of the issues cannot exceed* αJ'. The rule of $(1-\frac{\alpha}{2})$ relates to the Anscombe's $\alpha-$paradox: the proportion of voters who disagree with the majority outcome about more than half of the issues exceeds α.

We investigate in this paper an alternative condition sufficient for avoiding the Anscombe's $\alpha-$paradox, which relates to an alternative way to measure the level of unanimity among voters' opinions. Instead of referring to the average proportion of issue-wise agreements, we define the unanimity level of an electorate by considering a distance between potentially cast ballots.

To be more precise, we define a preference as a type of ballot, that is, a potential ballot which may be cast by one or several voters[2]. Furthermore, the level of unanimity in a set of preferences is defined as the maximal number of issues any two preferences agree on. Alternatively, the level of unanimity is the number of issues minus the maximal symmetric distance between two elements of this set (that is the number of issues which they disagree on). The larger this distance, the lower the level of unanimity that prevails among voters. This measure of unanimity allows for results that hold for any vote profile, that is any distribution of votes among preferences.

We compute for large electorates, and for any value of α in $]0,1[$ the exact value of the maximal distance between preferences under which, for any vote profile, the Anscombe's $\alpha-$paradox never holds. Moreover, we show that the value of the maximal distance between preferences provides an upper bound for the number of $\gamma-$controversial issues, defined as those for which the prevailing coalition comprises less than γ% of the voters.

The paper is organized as follows. Part 2 provides some notations and definitions. We present in Part 3 the already known conditions which allow to avoid the Anscombe's paradox, and show that the distance between preferences matters for the Anscombe's paradox to occur. Results are presented in Part 4. We conclude with several comments about our results and possible routes to further research.

2 The Anscombe's paradox

Given any two integers J and C, where J is odd, we consider a set of individuals $\mathcal{J}=\{1,...,J\}$ facing a set $\mathcal{C}$ of C distinct dichotomous issues. A *vote* is a vector $x=(x^1,...,x^C)\in\{0,1\}^C$, where $x^c=1$ (resp. 0) means that issue c is approved (resp. disapproved). For any vote x, we define $-x$ by: $\forall c$, $-x^c=1\Leftrightarrow x^c=0$. A *vote domain* is a subset $\varDelta$ of $\{0,1\}^C$. Each element $x\in\varDelta$ describing a vote that is potentially cast, a vote domain $\varDelta$ is a list of potential opinions that can be potentially defended within the society. Given a vote domain $\varDelta$, a *vote profile* is an element $X_\varDelta=(x_1,...,x_J)$ of $\{0,1\}^{CJ}$, where $x_j=(x_j^1,...,x_j^C)\in\{0,1\}^C$ is the vote of individual $j\in\mathcal{J}$, and where $x_j\in\varDelta$ for all j[3]. In order to lighten notations, we will write X instead of $X_\varDelta$ whenever no confusion is possible. Vote profiles will be represented by matrices $[x_j^c]_{j=1,...,J}^{c=1,...,C}$, where columns refer to individual votes. A vote domains are also represented by matrices $[x_h^c]_{h=1,...,H}^{c=1,...,C}$, where H is a natural number, and where $x_h\neq x_{h'}$ for all $h\neq h'$ in $\{1,...,H\}$.

The *issue-wise majority rule* (hereafter majority rule) is the function μ from $\{0,1\}^{CJ}$ to $\{0,1\}^C$ defined by $\mu(X)=(\mu^c(X))_{c\in\mathcal{C}}$, where $\forall c\in\mathcal{C}$, $\mu^c(X)=1\Leftrightarrow\sum_j x_j^c>\frac{J}{2}$. Hence, an issue is approved if and only if it receives more approvals than disapprovals.

The Anscombe's *paradox* holds at X if $\mid\{j:\mid\{c\in\mathcal{C}:x_j^c\neq\mu^c(X)\}\mid>\frac{C}{2}\}\mid>\frac{J}{2}$: more than one half of the voters disagree with the outcome of the majority rule on more than one half of the issues. The

[1] *'If N individuals cast yes-or-no votes on K proposals then, whatever the decision method employed to determine the outcomes of the votes on these proposals, if the average fraction of voters, across all proposals, comprising the prevailing coalitions is at least three-fourths, then the set of voters who disagree with a majority of the outcomes cannot comprise a majority'* (Wagner, 1983, pages 305-306).

[2] Table 1 above describes a case with 4 preferences.

[3] A possible interpretation is that a vote domain $\varDelta$ describes a political culture, whereas a vote profile $X_\varDelta$ describes a specific electorate whose members belong to the same culture.

Anscombe's paradox can equivalently be defined as follows. The *Hamming distance* between any two outcomes y and z in $\{0,1\}^C$ is defined by $d(y,z) =\mid \{c \in \mathcal{C} : y^c \neq z^c\} \mid$. Hence, the Anscombe's paradox holds at X if $\mid \{j : d(x_j, \mu(X)) > \frac{C}{2}\} \mid > \frac{J}{2}$.

Let $\mathcal{R}$ be the set of complete preorders of $\{0,1\}^C$. For any ballot x, the *Hamming preference from* x is the element $R(x) \in \mathcal{R}$ defined by: $\forall y, z \in \{0,1\}^C$, $(y,z) \in R(x)$ if $d(y,x) < d(z,x)$, that is, a voter having cast x prefer the outcome y than z if x disagrees with y on less issues than with z. It is straightforward to check that $R(x)$ is represented by the utility function U_x defined on $\{0,1\}^C$ by $U_x(y) = C - d(y,x)$. For any $y, z \in \{0,1\}^C$, we say that y defeats z in the vote profile $X = (x_1, ..., x_J)$ if $\mid \{j : d(y, x_j) \leq d(z, x_j)\} \mid > \mid \{j : d(z, x_j) \leq d(y, x_j)\} \mid$ (or, equivalently $\mid \{j : U_{x_j}(y) \geq U_{x_j}(z)\} \mid > \frac{J}{2}$).

Proposition 1 *The Anscombe's paradox holds at profile X if and only if $-\mu(X)$ defeats $\mu(X)$ in X.*

Proof Let $X = (x_1, ..., x_J)$ be such that $(-w(X))$ defeats $w(X))$. Since for any $y \in \{0,1\}^C$, $U_{x_j}(y) \geq \frac{C}{2} \Leftrightarrow U_{x_j}(y) \geq U_{x_j}(-y)$, then one get that $\mid \{j : U_{x_j}(-\mu(X)) \geq \frac{C}{2}\} \mid > \frac{J}{2}$. Hence, $\mid \{j : U_{x_j}(\mu(X)) \geq \frac{C}{2}\} \mid < \frac{J}{2}$, so that $\mid \{j : d(x_j, \mu(X)) > \frac{C}{2}\} \mid > \frac{J}{2}$, and the Anscombe's paradox holds at X. Conversely, if the paradox holds at X, then $\mid \{j : d(x_j, \mu(X)) > \frac{C}{2}\} \mid = \mid \{j : d(x_j, -\mu(X)) < \frac{C}{2}\} \mid > \frac{J}{2}$. which implies that $\mid \{j : U_{x_j}(-\mu(X)) \geq \frac{C}{2}\} \mid = \mid \{j : U_{x_j}(-\mu(X)) \geq U_{x_j}(\mu(X))\} \mid > \frac{J}{2}$. This ensures that $(-\mu(X))$ defeats $\mu(X))$ in X ♦

We consider a more general formulation of the Anscombe's paradox. Given $\alpha \in]0,1[$, *the Anscombe's α-paradox* holds at X when $\mid \{j : d(x_j, \mu(X)) > \frac{C}{2}\} \mid > \alpha J$: a proportion α of the voters disagrees with the majority outcome on a majority of issues.

What the Anscombe's $\alpha-$paradox states is that issue-wise majority voting may provide a poor compromise between diverging votes, where 'poor' relates to the value of α: the higher α, the higher the proportion of voters likely to complain, by forming a coalition in favor of the opposite of the majority rule outcome.

3 Does mutually close preferences matter?

We briefly review two conditions which are sufficient to avoid the paradox. The first, called *single-switchness*, relates to vote domains. Given a vote domain Δ, and a subset $\mathcal{D} \subseteq \mathcal{C}$ of issues, a $\mathcal{D}$-relabelling of Δ is obtained by reversing in each votes approvals and disapprovals regarding issues in $\mathcal{D}$. Furthermore, for any permutation σ of $\mathcal{C}$, a σ-permutation of Δ is the vote domain obtained by reshuffling the set of issues (i.e. columns of Δ) without modifying the voters' positions regarding each of them[4].

Two vote domains Δ and Δ' are *equivalent* if there exist a subset $\mathcal{D} \subseteq \mathcal{C}$ of issues and a permutation σ of $\mathcal{C}$, such that Δ' is obtained from Δ through a $\mathcal{D}$-relabelling combined with the σ-permutation of Δ. Furthermore, $\Delta = \{x_1, ..., x_H\}$ has a *single-switch representation* if in each vote x_h, there exists at most one issue $1 \leq c(h) \leq C-1$ such that $x_h^{c(h)} \neq x_h^{c(h)+1}$. Moreover, Δ is said to be *single-switch* if it

[4] The $\mathcal{D}-$*relabelling* of $\Delta = [x_h^c]_{c=1,\dots,C}^{h=1,\dots,H}$ is the domain $\Delta^{\mathcal{D}} = [y_h^c]_{c=1,\dots,C}^{h=1,\dots,H}$ defined by: $\forall c \in \mathcal{D}$, $\forall h$, $x_h^c = 1 \Leftrightarrow y_h^c = 0$, and $\forall c \notin \mathcal{D}$, $\forall h$, $x_h^c = y_h^c$. The σ-*permutation* of Δ is the vote domain $\Delta^\sigma = [x_h^{\sigma c}]_{c=1,\dots,C}^{h=1,\dots,H}$ defined by: $\forall c$, $\forall h$, $x_h^{\sigma c} = x_h^{\sigma^{-1}(c)}$.

is equivalent to a vote domain having a single-switch representation[5]. It is shown in Laffond and Lainé (2006) that, at any vote profile X_Δ built from a single-switch domain Δ, no outcome is more preferred by a majority of voters than $\mu(X_\Delta)$. Using Proposition 1, one thus have

Proposition 2 *(Laffond and Lainé, 2006) No vote profile built from a single-swith vote domain can face the Anscombe's paradox.*

Single-switchness is a strong restriction upon vote domains. It relates to some inter-vote consistency: indeed, a vote domain Δ is single-switch if and only if, for any two votes x and y in Δ, the set of issues approved in x either contains the set of those approved by y, or contains the set of those disapproved by y.

The second condition, called *the rule of* $(1-\frac{\alpha}{2})$, deals with the average level of consensus that prevails issue-wise in a vote profile. Given a vote profile X and an issue c, we call *agreement level* on c the number $a_c(X) = \mid \{j : x_j^c = \mu^c(X) \mid$, that is the number of individuals who agree with the majority will about c. Let $a(X) = \frac{1}{C}\sum_{c\in\mathcal{C}} a_c(X)$ be the average level of agreement across issues.

Proposition 3 *(Wagner, 1984) Consider any $\alpha \in]0,1[$. If X is a vote profile such that $a(X) \geq (1-\frac{\alpha}{2})J$, then X does not face the Anscombe's $\alpha-$paradox*[6]*.*

Hence, if the average size of majority among issues is at least $(1-\frac{\alpha}{2})$, no more than α voters disagree with the majority outcome on no more than half of the issues. In particular, the rule of three-fourth (Wagner, 1983) states that the paradox never holds at a vote profile where, on average, at least 75% of the voters agree on each of the issues.

We suggest here an alternative measure of unanimity related to vote domains, which is based on the maximum number of issues about which two potential votes disagree on. We denote $r_\Delta = Max\{\frac{d(x_h, x_{h'})}{C}, h, h' = 1, ..., H\}$ the maximal relative Hamming distance between two votes in the vote domain Δ. We first claim that the paradox may hold at a profile where votes differ on one half of the issues.

Proposition 4 *For any $\varepsilon > 0$, there exists a vote profile X such that $r_X < \frac{1}{2} + \varepsilon$ and the Anscombe's paradox holds at X.*

Proof Pick up an integer $k > 0$, and consider the vote profile X defined below, where $C = 2k+1$ and $J = 2k-1$:

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[5] For instance, the following vote domain Δ

	x_1	x_2	x_3	x_4
1	1	1	0	0
2	0	1	1	0
3	1	1	1	0
4	0	0	1	1

is single switch. To see why, the {1}-relabelling of Δ gives

	x_1	x_2	x_3	x_4
1	0	0	1	1
2	0	1	1	0
3	1	1	1	0
4	0	0	1	1

, while

	x_1	x_2	x_3	x_4
4	0	0	1	1
1	0	0	1	1
2	0	1	1	0
3	1	1	1	0

has a single-switch representation.

[6] Wagner (1984) proves in fact the following more general result: let α, $\beta \in]0,1[$. If X is such that $a(X) \geq (1-\alpha.\beta)J$, then the proportion of voters who disagree with $\mu(X)$ on more than $\beta\%$ of the issues will be less than α.

TABLE 2

	1	2	...	$k-1$	k	$k+1,...2k-1$
1	1	0	...	0	0	0
2	0	1	...	0	0	0
...	...	...	...	...	...	...
$k-1$	0	0	...	1	0	0
k	0	0	...	0	1	0
$k+1$	0	1	...	1	1	0
$k+2$	1	0	...	1	1	0
...	...	...	...	...	...	...
$2k-1$	1	1	...	0	1	0
$2k$	1	1	...	1	0	0
$2k+1$	1	1	...	1	0	0

It is obviously seen that $\mu(X) = (0, ..., 0)$. Moreover, $\forall j, h \leq k$, $d(x_j, x_h) = 4$, $\forall j < k$ and $\forall h > k$, $d(x_j, x_h) = k+1$, $\forall h > k$, $d(x_k, x_h) = k$, and $\forall j, h > k$, $d(x_j, x_h) = 0$. Thus, $r_X = \frac{k+1}{2k+1}$. Furthermore, $\forall j \leq k$, $U_j(\mu(X)) = k$, while $U_j(-\mu(X)) = k+1$. Hence, the paradox prevails. The conclusion follows from the fact that $Lim_{k\to\infty} \frac{k+1}{2k+1} = \frac{1}{2}$ ♦

However, the paradox never holds at a vote profile involving votes that mutually differ on less than a third of the issues.

Proposition 5 *Let X be any vote profile such that $r_X \leq \frac{1}{3}$. Then the Anscombe's paradox does not hold at X.*

Proof Note first that, through an appropriate relabelling of issues, one can assume that $\mu(X) = (0, ..., 0)$. For any voter j, let $\delta_j = \sum_c x_j^c$. From the definition of r_X, one get that: $\forall j \in \mathcal{J}$, $\sum_{h\neq j} d(x_j, x_h) \leq r_X C(J-1)$.

Furthermore, it is well-known that, given any vote profile X, $\mu(X)$ maximizes $\sum_{j\in\mathcal{J}} U_{x_j}$ in the set $\{0,1\}^C$ of all possible outcomes (see e.g. Brams, Kilgour and Sanver (2007) for a formal proof). Hence, one get that $\sum_j d(x_j, \mu(X)) \leq r_X C(J-1)$. Moreover, it follows from the definition of $\mu(X)$ that:

- $\forall j \in \mathcal{J}$, $d(x_j, \mu(X)) = \delta_j$ (1)

Hence, one get that:

- $\sum_j \delta_j \leq r_X C(J-1)$ (2)

Now, suppose that the paradox holds at X. Let $A = \{j : \delta_j > \frac{C}{2}\}$. It follows that $| A | > \frac{J}{2}$. For any j, let $1(j) = \{c : x_j^c = 1\}$ and $0(j) = \mathcal{C} - 1(j)$. Furthermore, let consider any two voters j and h. Since $d(x_j, x_h) \leq r_X C$, then $| 0(j) \cap 1(h) | + | 1(j) \cap 0(h) | \leq r_X C$. This immediately implies that:

- $\delta_j - r_X C \leq \delta_h \leq \delta_j + r_X C$
- $\delta_h - r_X C \leq \delta_j \leq \delta_h + r_X C$

Since $\delta_j > \frac{C}{2}$ for all $j \in A$, it follows from the two inequalities above that:

- $\forall j \notin A$, $\delta_j \geq \frac{C}{2} - r_X C$ (3)

Combining (1), (2) and (3) leads to:

- $\frac{C}{2}\frac{J}{2} + \frac{J}{2}(\frac{C}{2} - r_X C) \leq \sum_j \delta_j < r_X CJ$

Thus, $CJ\frac{1-r_X}{2} < r_X CJ$, which proves that $r_X > \frac{1}{3}$ ♦

Propositions 4 and 5 confirm the intuition: if votes are mutually close enough (differ on less than a third of the issues), the outcome of majority rule cannot be less preferred than its opposite by a majority of voters. And if votes are distant enough (one half of the issues), the Anscombe's paradox may prevail. We are left with the range of possible distance values. Our main result consists in computing the exact bound of the maximal distance between votes in a vote domain under which the α-paradox holds at no profile built from this domain.

The maximal relative distance between votes in some vote profile X clearly implies nothing about the average agreement level in X. For instance, consider the profile described in Table 3 below:

TABLE 3

x_1	x_2	x_3	x_4
1	1	0	1
1	0	1	1
1	1	1	1
1	1	1	1
1	1	1	0
0	1	1	1
0	0	0	1
0	1	0	0
0	0	0	0

Then, $r_X = \frac{1}{3}$, and thus there is no paradox. Furthermore, the average level of issue-wise consensus is low, since $a(X) = 0.55$. However, we show below the existence of a relationship between r_X and the number of issues for which a given level of consensus prevails.

4 Results

Given a vote profile X, define $m_X =\mid \{j \in \mathcal{J} : U_j(-\mu(X)) > U_j(\mu(X))\} \mid$ and $p_X = \frac{m_X}{J}$. Hence, m_X (resp. p_X) is the number (resp. proportion) of voters who less prefer the majority outcome than its opposite. Moreover, let $n_X = (J - m_X)$ the number of voters having the reverse opinion. A vote profile X is said to be *large* if both m_X and n_X are large enough to approximate both $\frac{n-1}{n}$ and $\frac{m-1}{m}$ by 1. We address the following question: what is the maximal value $r(\alpha)$ of the maximal distance between two votes under which no large vote profile faces the $\alpha-$paradox? Equivalently, we solve the following problem:

Let $\alpha \in]0,1[$. *Find the minimal value* $r(\alpha)$ of $r \in [0,1]$ *for which there exists a large profile* X *such that* $r_X = r$ *and* $p_X > \alpha$.

It follows from Propositions 4 and 5 that $r(\frac{1}{2}) \in [\frac{1}{3}, \frac{1}{2}]$. Our first theorem (proven in Appendix B) specifies the function $r(\alpha)$.

Theorem 1 $r(\alpha) = \frac{1}{4(1-\alpha)}$ *if* $\alpha \leq \frac{1}{4}$, *and* $r(\alpha) = \frac{\sqrt{\alpha}-\alpha}{1-\alpha}$ *if* $\frac{1}{4} \leq \alpha \leq 1$.

Figure 1 in Appendix A depicts the function $r(\alpha)$. The values of α appear on the $x-$axis. The Anscombe paradox occurs for all values of α above $\frac{1}{2}$. Since $r(\frac{1}{2}) \approx 0.414$, then the Anscombe's paradox cannot hold when the relative distance between any two votes is less than 41.4%. Furthermore, the Anscombe's $(0.2)-$paradox never holds if no two ballots differ on more than 31.25% of the issues: in such a case, always less than 20% of the voters are put on the losing side on more than half of the issues. Similarly, if any two votes disagree on less than 47.21% of the issues, always less than 80% of the electorate will disagree with the majority outcome on more than half of the issues.

Theorem 1 is very different in spirit from the rule of $(1 - \frac{\alpha}{2})$. Indeed, the criterion of the maximal distance applies to vote domain, and thus is independent from the distribution of the voters among ballots. Put differently, consider any vote domain $\Delta = \{x_1, ..., x_H\}$ of different ballots such that the distance between any two votes is less that $r(\alpha)$. Then, Theorem 1 states that, whatever the number of voters casting each of the x_h in Δ, that is, whatever the vote profile X built from Δ, X cannot face the Anscombe's $\alpha-$paradox. We interpreted above vote domains as sets of admissible opinions that can be defended in the society, independently from the actual distribution of votes among them. Hence, the maximal distance criterion should be seen as an ex-ante measure of unanimity. Such is not the case for the rule of $(1 - \frac{\alpha}{2})$, which explicitly relates to the distribution of votes: if, on average, all issues show a sufficient level of agreement among the *actual* votes, then the paradox is impossible.

Despite this important difference, there is still a link between our measure of unanimity and the number of issues which do not reach a high level of agreement. Given a vote profile X and an issue c, let $m^1_c(X)$ (resp. $m^0_c(X)$) stand for the number of voters who approve (resp. disapprove) c. We define $s(X,c) = \frac{|m^1_c(X) - m^0_c(X,c)|}{J} = \frac{|2a_c(X) - J|}{J}$, that is the relative majority margin for c. If $s(X,c)$ is close to 1, then almost all voters share the same opinion on c, so that c does not appear as controversial: opinions are close to the unanimity about c. Similarly, if $s(X,c)$ is close to 0, then there an almost tie between the two opinions. Hence, $s(X,c)$ is to be interpreted as a measure of the level of unanimity regarding issue c. Let $\gamma \in [0,1]$. An issue c is $\gamma-$controversial in X if $s(X,c) \leq \gamma$. Given a vote profile X, The proportion of $\gamma-$controversial issues is denoted by $J(\gamma, X)$.

Theorem 2, proven in the Appendix C, confirms the intuition that the closer the ballots, the fewer controversial candidates.

Theorem 2 *Let Δ be a vote domain such that $d(x,y) \leq r$ for all $x,y \in \Delta$. Then, in any large vote profile X_Δ, $J(\gamma, X) \leq \frac{2r}{1-\gamma^2}$.*

As an illustration, assume that $\gamma = 0.2$. Thus, an issue is controversial when reaching a relative majority margin of at most 40%, that is level of agreement of 60%. Then, if no two votes disagree on more than 20% of the issues, at most 41.6% of them are controversial. Similarly, if an issue is controversial if it does not reach an agreement of 75% among voters ($\gamma = 0.5$), then less than 53.33% of the issues are controversial.

5 Discussion

We establish in this paper the exact relationship $r(\alpha)$ which links the level of unanimity in preferences, defined as the maximal distance between ballots, and the existence of a generalized Anscombe's paradox, called the Anscombe's $\alpha-$paradox, where more than a given proportion α of the voters disagree on a majority of issues with the outcome of issue-wise majority voting. As already shown by the three-fourths rule, the relationship between the occurence of the paradox and closedness of preferences is quite intuitive. Our main contribution is to give an alternative exact answer to the question of such relationship, based on a specific measure of consensus, or unanimity, within a set of *potential votes* (rather than a set of actual votes). This approach is along the lines of preference domain restriction that have been extensively studied by Social Choice Theory. A potential vote is defined as a ballot seen as admissible from a cultural or political viewpoint, a vote profile being a specific distribution of votes among admissible ballots. This distinction between a vote domain, defined as a set of potential votes, and a vote profile allows for deriving properties that hold for *any* profile, that is any distribution of votes. In particular, the function $r(\alpha)$ relates to vote domains, and contrasts with the three-fourth rule, which is a property of vote profiles. As a consequence, the measure of unanimity in a vote domain should not be depending on the distribution of votes. As a distance-based criterion, the maximal distance follows a very conservative approach, by implicitly assuming that any vote distribution is possible, including the one that implements a highly polarized political landscape, where the electorate equally splits between two ballots that are the most distant to each other. Considering alternative measures of closedness within a vote domain calls for further research. Furthermore, a complementary worthwhile analysis would focus on unanimity measures in vote profiles, along the lines drawn in the studies on consensus measures in preference profiles (Kendall (1955), Hays (1960), Kemeny and Snell (1962), Cook and Seiford (1978), (1982), Bosch (2005), Garcia-Lapresta and Perez-Roman (2008), (2009), Alcade-Unzu and Vorsatz (2010)). However, note that the framework of multiple referendum does not consider preferences in the usual sense (that is, defined as linear or weak orders), but judgment sets[7].

The function $r(\alpha)$ shows that escaping from the $\alpha-$paradox in all vote profiles requires to drastically restrict the set of possible votes. For instance, we know that avoiding the Anscombe's paradox ($\alpha = 0.5$) with 5 issues imposes a maximal distance 2 between any two votes. It is easily seen that the maximal number of possible votes under this constraint is 5 out of 32 possible votes. More generally, ... An alternative way to illustrate the level of restriction brought by the function $r(\alpha)$ is to count the maximal

[7] To our knowledge, distance-based measures of consensus among judgment sets remain to be studied. An interesting step is offered by Duddy and Piggins (2010), who characterize a distance between two judgment sets that differs from the Hamming distance. See List and Puppe (2009) for a recent survey on the aggregation of judgment sets.

number of possible votes sharing a common structure. For example, suppose that all issues can be ordered along a commonly perceived left-right political spectrum (say $1 \prec 2 \prec ... \prec C$). A vote x is single-plateaued if there exist two issues $c_1(x), c_2(x)$ such that $x^c = 1 \Leftrightarrow c \in \{c_1(x), ..., c_2(x)\}$. Assume that all votes are single-plateaued, and that $c_2(x) - c_1(x)$ is the same for all votes. If r denotes the maximal possible distance between two votes, one can allow for at most $\frac{r}{2} + 1$ if r is even, and $\frac{r-1}{2} + 1$ if r is odd.

We excluded the case where $\alpha = 1$ from the analysis (where all voters are on the losing side on a majority of issues). Indeed, as already mentioned in the proof of Proposition 5, we have that, given any vote profile $X = (x_1, ..., x_J) \in \{0,1\}^{CJ}$, $\mu(X)$ minimizes $\sum_{j=1}^{J} d(x, x_j)$ on $\{0,1\}^C$ (see e.g. Brams, Kilgour and Sanver (2007) for a formal proof). This implies that issue-wise majority voting is efficient when individual preferences over programs based on the Hamming distance criterion: given a vote $x \in \{0,1\}^C$, the utility derived from the program $y \in \{0,1\}^C$ is defined by $U_x(y) = C - d(x, y)$. Hence, efficiency holds whatever the unanimity level within the vote profile[8].

The $\alpha-$paradox implicitly assumes that voter are dissatisfied with an outcome that differs from their ballots on a majority of issues. A natural question is to investigate how the function $r(\alpha)$ evolves for alternative measures of dissatisfaction. This leads to the following problem: say that a voter is $\delta-$dissatisfied with the outcome x if the relative Hamming distance between x and her ballot is at least $\delta \in]0,1]$; let $\alpha \in [0,1]$; find the minimal value $r(\alpha, \delta)$ of $r \in [0,1]$ for which there exists a vote profile X where the maximal distance between two votes is r and $\alpha\%$ of the voters are $\delta-$dissatisfied.

Finally, our results can be offered an alternative dual interpretation[9]. Define the parameter α as a measure of social acceptability of the referendum outcome. Then the function $r(\alpha)$ provides the level of consensus that must a priori prevail within the society in order for issue-wise majority voting to reach a socially acceptable outcome for any distribution of votes. Following this approach in more general collective choice situations may allow to compare different voting rules according to the level of consensus they require for their outcome to share some property defining social acceptability.

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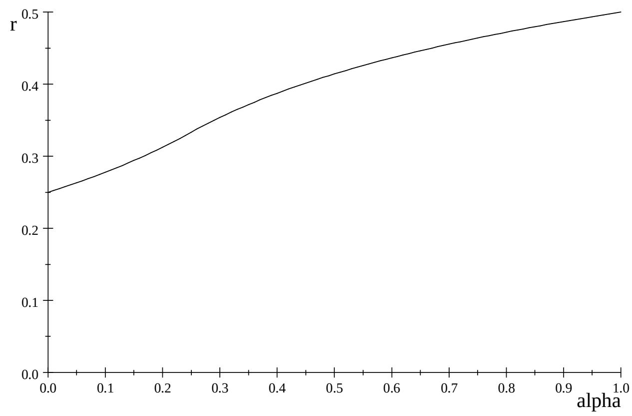
[8] The reader will find in Çuhadaroğlu and Lainé (2010) a study of the efficiency of majority voting in multiple referendum under alternative distance-based preferences.

[9] We thank one of the reviewers for having suggested this route for further research.

7 Appendix A: Figure 1, graph of $r(\alpha)$



8 Appendix B: proof of Theorem 1

8.1 Preliminaries

Let $X \in \{0,1\}^{CJ}$ such that $\forall j,h \in \mathcal{J}, \frac{d(x_j,x_h)}{C} \leq r$. Let $N(X) = \left\{j \in \mathcal{J}, \frac{d(x_j,\mu(X))}{C} < 1/2\right\}$ (with cardinality $n = n(X)$) be the set of individuals who prefer the outcome of the majority rule than its opposite and $M(X) = \mathcal{J} - N(X)$ ($m = m(X) = J - n$).

The proof is organized in several steps. The first one provides a useful technical tool for studying vote profiles. The purpose here is to show that one can build from X another vote profile $X' = (x'_1, ..., x'_J) \in \{0,1\}^{C'J}$ where

- the distances between individuals does not exceed r: $\forall j,h \in \mathcal{J}, \frac{d(x'_j,x'_h)}{C'} \leq r$.
- $N(X') = N(X) = N$ and $M(X') = M(X) = M$: the way individuals compare the majority outcome and its opposite does not change from X to X'.
- all distances between two ballots cast by individuals who prefer the majority outcome $\mu(X')$ are the same :$\forall j,h,j',h' \in M$, $d(x'_j,x'_h) = d(x'_{j'},x'_{h'})$
- all distances between two ballots cast by individuals who prefer the opposite of the majority outcome $-\mu(X')$ are the same: $\forall j,h,j',h' \in N$, $d(x'_j,x'_h) = d(x'_{j'},x'_{h'})$
- all distances between two ballots cast by individuals who disagree when comparing $\mu(X')$ and $-\mu(X')$ are the same: $\forall j,h \in M$, $\forall j',h' \in N$, $d(x'_j,x'_{j'}) = d(x'_h,x'_{h'})$

8.2 Block (k,l)-matrices

We first introduce the notion of block matrix. A matrix with 0 and 1 entries involving J columns is a block matrix if defined as follows: given a partition of the J columns into two sets M and N with respective sizes m and n, all row contain k (resp. l) entries 1 in restriction to M (resp. to N) ; and the number of rows is equal to the number of possible joint permutations of both sets M and N. Formally, let $0 \leq k \leq m$, $0 \leq l \leq n$. The cardinality of the set $Z_{k,l} = \left\{ z \in \{0,1\}^J, \sum_{j \in M} z_j = k, \sum_{j \in N} z_j = l \right\}$ is $\binom{k}{m}\binom{l}{n}$.

A *block* $(k,l)-$*matrix* a is matrix $Z(k,l)=(z_1,...,z_J)$ with J columns and $\binom{k}{m}\binom{l}{n}$ rows, and whose rows are all the elements of $Z_{k,l}$. One easily computes the distances between any two of columns of $Z(k,l)$:

(1) if $j,h\in M$, $d(z_j,z_h)=2\binom{k-1}{m-2}\binom{l}{n}$

(2) if $j,h\in N$, $d(z_j,z_h)=2\binom{k}{m}\binom{l-1}{n-2}$

(3) if $h\in M$, $j\in N$, $d(z_h,z_j)=\binom{k-1}{m-1}\binom{l}{n-1}+\binom{k}{m-1}\binom{l-1}{n-1}$

(4) if $j\in M$, $d(z_j,\mu(Z))=\begin{cases}\binom{k}{m-1}\binom{l}{n} \text{ if } k+l>J/2,\\ \binom{k-1}{m-1}\binom{l}{n} \text{ if } k+l<J/2\end{cases}$

(5) if $j\in N$, $d(z_j,\mu(Z))=\begin{cases}\binom{k}{m}\binom{l}{n-1} \text{ if } k+l>J/2,\\ \binom{k}{m}\binom{l-1}{n-1} \text{ if } k+l<J/2\end{cases}$

8.3 Composition of matrices

Define $\sum_1=\{\sigma_1,...,\sigma_{m!n!}\}$ as the set of permutations of $\mathcal{J}$ such that $\sigma(M)=M$ and $\sigma(N)=N$. Given $\sigma\in\sum_1$, the matrix $X(\sigma)\in\{0,1\}^{CJ}$ is defined by: $\forall c,j,\ X(\sigma)^c_j=X^c_{\sigma(j)}$. The number of such matrices is thus $m!n!$.

The composition ϕ of two matrices is defined in the following way : let A,B,C be three matrices whose elements are $\{0,1\}$, with J columns and respectively a, b, and $a+b$ rows. We say that $C=\phi(A,B)$ if $C^k_j=A^k_j$ when $k\leq a$, and $C^k_j=B^{k-a}_j$ when $k\geq a$. This simply means that C is obtained by putting "A over B".

Notice that if $\frac{d(A_j,A_h)}{a}\leq r$ and $\frac{d(B_j,B_h)}{b}\leq r$, then $\frac{d(C_j,C_h)}{a+b}\leq r$.

Then we can define

$X_1=X(\sigma_1)$, $X_2=\phi(X_1,X(\sigma_2))$, and, for every $3\leq h\leq m!n!$, $X_h=\phi(X_{h-1},X(\sigma_h))$.

This construction leads to the matrix $X'=(x'_1,...,x'_J)$, which involves J columns and $m!n!C$ rows. We claim that X' shares the 5 properties above.

First, we have $\forall j,h\in\mathcal{J}$, $\frac{d(x'_j,x'_h)}{m!n!C}\leq r$: indeed, this property is stable under composition. Second, we have $N(X')=\left\{j\in\mathcal{J}:\frac{d(x'_j,\mu(X'))}{m!n!C}<1/2\right\}=N(X)$, since $N(X)$ is stable under any permutation $\sigma\in\sum_1$. Thus X and X' share the same properties relatively to r.

Consider a row x^c in X. Since x^c generates one new specific row in each matrix $X(\sigma)$, then x^c generates $m!n!$ rows in X', each being a permutation of x^c. Furthermore, if x^c contains k entries 1 in M and l entries 1 in N, then X' contains $\binom{k}{m}\binom{l}{n}$ different rows induced by x^1, which define a block $(k.l)-$matrix. It follows from symmetry that x^c generates $\frac{m!n!}{\binom{k}{m}\binom{l}{n}}$ block $(k.l)-$matrices contained in X'. Let $c(k.l)$ be the number of rows x^c in X such that $k=\sum\limits_{j\in M}x^c_j$, $l=\sum\limits_{j\in N}x^c_j$. One get that X' is a collection of $\frac{m!n!}{\binom{k}{m}\binom{l}{n}}c(k,l)$ block $(k,l)-$matrices.

Using the above expressions of the distance between columns in some block $(k,l)-$matrix Z, the contribution of all block $(k,l)-$matrices to the distance between any two ballots (i.e. columns of X'), as well the distance between each ballot and the majority outcome is given by:

(1) if $j,h\in M$, $d(x'_j,x'_h)=\frac{2m!n!\binom{k-1}{m-2}\binom{l}{n}}{\binom{k}{m}\binom{l}{n}}c(k,l)$

(2) if $j,h\in N$, $d(x'_j,x'_h)=\frac{2m!n!\binom{k}{m}\binom{l-1}{n-2}}{\binom{k}{m}\binom{l}{n}}c(k,l)$

(3) if $h\in M$ and $j\in N$, $d(x'_j,x'_h)=[\binom{k-1}{m-1}\binom{l}{n-1}+\binom{k}{m-1}\binom{l-1}{n-1}]\frac{m!n!}{\binom{k}{m}\binom{l}{n}}c(k,l)$

(4) if $j\in M, d(x'_j,\mu(X'))=\begin{cases}\frac{m!n!\binom{k}{m-1}\binom{l}{n}}{\binom{k}{m}\binom{l}{n}}c(k,l) \text{ if } k+l>J/2,\\ \frac{m!n!\binom{k-1}{m-1}\binom{l}{n}}{\binom{k}{m}\binom{l}{n}}c(k,l) \text{ if } k+l<J/2\end{cases}$

(5) if $j \in N, d(x'_j, \mu(X')) = \begin{cases} \frac{m!n!\binom{k}{m}\binom{l}{n-1}}{\binom{k}{m}\binom{l}{n}} c(k,l) \text{ if } k+l > J/2, \\ \frac{m!n!\binom{k}{m}\binom{l-1}{n-1}}{\binom{k}{m}\binom{l}{n}} c(k,l) \text{ if } k+l < J/2 \end{cases}$

These can be simplified to :

(1) if $j, h \in M$, $d(x'_j, x'_h) = 2\frac{k(m-k)}{m(m-1)} m!n!c(k,l)$

(2) if $j, h \in N$, $d(x'_j, x'_h) = 2\frac{l(n-l)}{n(n-1)} m!n!c(k,l)$

(3) if $h \in M$, $j \in N$, $d(x'_j, x'_h) = \frac{(m-k)l+k(n-l)}{mn} m!n!c(k,l)$

(4) if $j \in M$, $d(x'_j, \mu(X')) = \begin{cases} \frac{m-k}{m} m!n!c(k,l) \text{ if } k+l > J/2, \\ \frac{k}{m} m!n!c(k,l) \text{ if } k+l < J/2 \end{cases}$

(5) if $j \in N$, $d(x'_j, \mu(X')) = \begin{cases} \frac{n-l}{n} m!n!c(k,l) \text{ if } k+l > J/2, \\ \frac{l}{n} m!n!c(k,l) \text{ if } k+l < J/2 \end{cases}$

Hence, since this contribution is the same across individuals, then X' fulfills all the properties mentioned above.

We are now ready to set the optimization program to be solved in a tractable form:

8.4 The optimization program

In the sequel and without loss of generality we restrict our analysis to the case of where X is a composition of block matrices, where Y is the initial matrix containing C' rows. Hence, X contains $C = m!n!C'$ rows. Let $I = \{i = (k_i, l_i) : 0 \leq k_i \leq m,\ 0 \leq l_i \leq n\}$ be the set of blocks $(k,l)-$matrices composed by X. For each i, let $c(k_i, l_i)$ be the number of rows in Y which generate a block $(k_i, l_i)-$matrix, and let $\theta_i = \frac{c(k_i,l_i)}{C'}$ be the proportion of those rows in Y.

It appears that all distances between ballots, as well as the way voters compare $\mu(X)$ and its opposite, can be written by means of weighted averages, where the weights are given by the θ_i, $i \in I$.

For each $i \in I$, let $u_i = \frac{k_i}{m} - \frac{1}{2}$ and $v_i = \frac{l_i}{n} - \frac{1}{2}$. With these notations, the different distances become:

(1) if $j, h \in M$, $\frac{d(x_j, x_h)}{C} = 2\sum_i \frac{m}{m-1}(\frac{1}{4} - u_i^2)\theta_i \leq r$

(2) if $j, h \in N$, $\frac{d(x_j, x_h)}{C} = 2\sum_i \frac{n}{n-1}(\frac{1}{4} - v_i^2)\theta_i \leq r$

(3) if $j \in M$, $h \in N$, $\frac{d(x_j, x_h)}{C} = 2\sum_i (\frac{1}{4} - u_i v_i)\theta_i \leq r$

Through a relevant relabelling of the issues, we can assume that 0 is the issue-wise majority will for each of the issues. Thus, for any block $(k_i, l_i) \in I$, we have $k_i + l_i < J/2$, and the distances between ballots and the majority outcome are given by :

(4) if $j \in M$, $\frac{d(x_j, \mu(X))}{C} = \sum_i (\frac{1}{2} + u_i)\theta_i \geq \frac{1}{2}$

(5) if $j \in N$, $\frac{d(x_j, \mu(X))}{C} = \sum_i (\frac{1}{2} + v_i)\theta_i \leq \frac{1}{2}$

Moreover, since 0 is the majority will for each issue:

(6) $mu_i + nv_i \leq 0$

Hence, the problem to be solved is finding the weights θ_i of the different blocks (u_i, v_i) in order to minimize r under the five preceding constraints. In other words, the problem writes:

minimize r

$$(S)\begin{cases}(1)\ \theta_i \geq 0, -\frac{1}{2} \leq u_i \leq \frac{1}{2}, -\frac{1}{2} \leq v_i \leq \frac{1}{2}, \sum_i \theta_i = 1\\ (2)\ 2\sum_i u_i^2\theta_i \geq \frac{1}{2} - \frac{m-1}{m}r\\ (3)\ 2\sum_i v_i^2\theta_i \geq \frac{1}{2} - \frac{n-1}{n}r\\ (4)\ 2\sum_i u_iv_i\theta_i \geq \frac{1}{2} - r\\ (5)\ mu_i + nv_i \leq 0\\ (6)\ \sum_i u_i\theta_i \geq 0\\ (7)\ \sum_i v_i\theta_i \leq 0\end{cases}$$

We can exploit the convexity properties of this non linear program to reduce its dimension.

8.5 Reducing the dimension of (S)

Constraint (5) implies that one cannot have $u_i > 0$ and $v_i > 0$. In fact, if $\{(u_i, v_i, \theta_i),\ i = 1, ..., I\}$ solves (S), then we can find another set of parameters values $\left\{(u\prime_i, v_i', \theta_i'), i = 1..., I\right\}$ which also solves (S), and where $v_i' \leq 0$ for any i. Indeed, take $\{(u_i, v_i, \theta_i), i = 1, ..., I\}$ that solves (S) and, for any i, define $u_i' = u_i$, $v_i' = -|v_i|$, and $\theta_i' = \theta_i$. Every constraint but (4) remains fulfilled with the new values of the parameters. If $u_i \leq 0$, then $u_i'v_i' \geq u_iv_i$. If $u_i > 0$, then $v_i' = v_i$ and $u_i'v_i' = u_iv_i$. It follows that $2\sum_i u_i'v_i'\theta_i \geq 2\sum_i u_iv_i\theta_i \geq \frac{1}{2} - r$.

This allows to delete constraint (7) and to transform constraint (1) into:

(1') $\theta_i \geq 0, -\frac{1}{2} \leq u_i \leq \frac{1}{2}, -\frac{1}{2} \leq v_i \leq 0, \sum_i \theta_i = 1$

All the constraints but (2) and (3) are linear in u_i and linear in v_i.

We know that if $0 \leq \alpha \leq 1$, then $(\alpha x + (1-\alpha)y)^2 \leq \alpha x^2 + \alpha y^2$. This allows to drastically reduce the dimension of the problem. Indeed, we represent below the set of the (u_i, v_i) values satisfying constraints (1') and (5): in both Figures 2 and 3, (1') requires that all (u, v) must be such that $v < 0$, whereas (5) restricts the (u, v) below the segment 0D ; when $m < n$, then D lies above C (Figure 2), while D is at the left of C when $m > n$ (Figure 3).

Figure 2

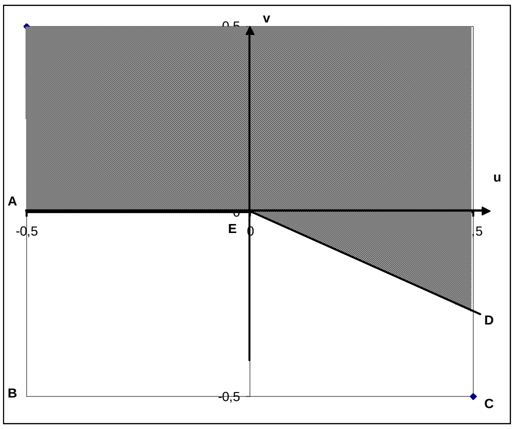
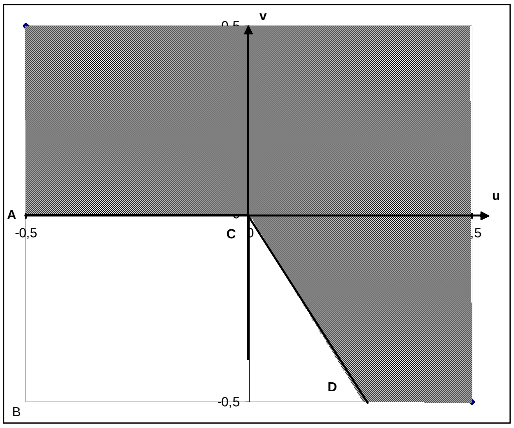


Figure 3



$\{(u_i, v_i, \theta_i), i = 1, ..., I\}$ is such that (S) is true, and, for example $u_1 < 0$ and $v_1 < 0$.Then we can substitute to the point (u_1, v_1, θ_1) two points $\left\{ \begin{array}{l} (0, v_1, \theta_1^0) \\ (-1/2, v_1, \theta_1^1) \end{array} \right.$, where θ_1^0 and θ_1^1 are such that $\left\{ \begin{array}{l} \theta_1^0 + \theta_1^1 = \theta_1 \\ \theta_1^0(0) + \theta_1^1(-\frac{1}{2}) = \theta_1 u_1 \end{array} \right.$.

We just have to check whether constraint (2) is fulfilled with this new set of parameter values. And this is easily done since we know from the result above that $\theta_1^0(0)^2 + \theta_1^1(-\frac{1}{2})^2 \geq \theta_1 u_1^2$. Hence, the solution of (S) must be belong to "the frontier of the admissible domain". We then have to address the two following cases:

8.5.1 Case 1 : $m < n$ In this case, we know (see Figure 2) that the solution consists of the three points $A(-\frac{1}{2}, 0, \theta_A)$, $B(-\frac{1}{2}, -\frac{1}{2}, \theta_B)$, $C(\frac{1}{2}, -\frac{1}{2}, \theta_C)$ and of some points between E and D $((u_i, -\frac{m}{n}u_i, \theta_i), 1 \leq i \leq I$ with $0 \leq u_i \leq \frac{1}{2})$

The constraints are rewritten :

(1) $\theta_A + \theta_B + \theta_C + \sum_i \theta_i = 1, 0 \leq u_i \leq \frac{1}{2}$

(2) $\frac{1}{2}(\theta_A + \theta_B + \theta_C) + 2\sum_i u_i^2\theta_i \geq \frac{1}{2} - \frac{m-1}{m}r$

(3) $\frac{1}{2}(\theta_B + \theta_C) + 2(\frac{m}{n})^2\sum_i u_i^2\theta_i \geq \frac{1}{2} - \frac{n-1}{n}r$

(4) $\frac{1}{2}(\theta_B - \theta_C) - 2\frac{m}{n}\sum_i u_i^2\theta_i \geq \frac{1}{2} - r$

(6)$\frac{1}{2}(-\theta_A - \theta_B + \theta_c) + \sum_i u_i\theta_i \geq 0$

If we replace in these inequalities θ_A and θ_B respectively by $\theta'_A = 0$ and $\theta'_B = \theta_A + \theta_B$, they remain true. Hence, we assume below that $\theta_A = 0$.

Consider the sign of the constraints. One checks that (4) implies both (2) and (3), except for the ratios $\frac{m-1}{m}$ and $\frac{n-1}{n}$. However, when $\sum_i u_i^2\theta_i$ is strictly positive and both m and n are large enough, both constraints (2) and (3) are fulfilled if (4) holds Thus the problem reduces to:

minimize r

(1) $\theta_B + \theta_C + \sum_i \theta_i = 1, 0 \leq u_i \leq \frac{1}{2}$

(4) $\frac{1}{2}(\theta_B - \theta_C) - 2\frac{m}{n}\sum_i u_i^2\theta_i \geq \frac{1}{2} - r$

(6)$\frac{1}{2}(-\theta_B+\theta_c)+\sum_i u_i\theta_i \geq 0$

and we will check at the end that $\sum_i u_i^2\theta_i > 0$.

Since $\frac{1}{\sum_i \theta_i}\sum_i u_i^2\theta_i \geq (\frac{1}{\sum_i \theta_i}\sum_i \theta_i u_i)^2$,we can substitute in (4) all the (u_i,θ_i) by their mean (u,θ_u) with $\theta_u = \sum_i \theta_i$. Finally, the problem is to find 4 numbers $\theta_B, \theta_C, \theta_u, u$ which minimize r under the constraints :

(1) $\theta_B, \theta_C, \theta_u \geq 0, \theta_B+\theta_C+\theta_u = 1, 0 \leq u \leq \frac{1}{2}$

(4) $\frac{1}{2}(\theta_B-\theta_C) - 2\frac{m}{n}u^2\theta_u \geq \frac{1}{2} - r$

(6)$\frac{1}{2}(-\theta_B+\theta_c)+\theta_u u \geq 0$

8.5.2 case 2 : $m > n$ In this case, we know (see Figure 3) that the solution consists of the two points $A(-\frac{1}{2},0,\theta_A)$,$B(-\frac{1}{2},-\frac{1}{2},\theta_B)$, and of some points between C and D $((-\frac{n}{m}v_i, v_i, \theta_i), 1 \leq i \leq I$ with $-\frac{1}{2} \leq v_i \leq 0)$

The constraints are rewritten :

(1) $\theta_A+\theta_B+\sum_i \theta_i = 1, -\frac{1}{2} \leq v_i \leq 0$

(2) $\frac{1}{2}(\theta_A+\theta_B) + 2(\frac{n}{m})^2\sum_i v_i^2\theta_i \geq \frac{1}{2} - \frac{m-1}{m}r$

(3) $\frac{1}{2}\theta_B + 2\sum_i v_i^2\theta_i \geq \frac{1}{2} - \frac{n-1}{n}r$

(4) $\frac{1}{2}\theta_B - 2\frac{n}{m}\sum_i v_i^2\theta_i \geq \frac{1}{2} - r$

(6)$\frac{1}{2}(-\theta_A-\theta_B) - \frac{n}{m}\sum_i v_i\theta_i \geq 0$

Again, if we substitute in these inequalities θ_A and θ_B by $\theta'_A = 0$ and $\theta'_B = \theta_A+\theta_B$, they remain true. Hence in the following, we shall assume that $\theta_A = 0$.

As in the previous case, using $\sum_i v_i^2\theta_i > 0$, when m and n are large enough both constraints (2) and (3) are fulfilled if (4) holds. Thus the problem reduces to

minimize r

(1) $\theta_B+\sum_i \theta_i = 1, -\frac{1}{2} \leq v_i \leq 0$

(4) $\frac{1}{2}\theta_B - 2\frac{n}{m}\sum_i v_i^2\theta_i \geq \frac{1}{2} - r$

(6)$-\frac{1}{2}\theta_B - \frac{n}{m}\sum_i v_i\theta_i \geq 0$

Since $\frac{\sum_i v_i^2\theta_i}{\sum_i \theta_i} \geq \left(\frac{\sum_i \theta_i v_i}{\sum_i \theta_i}\right)^2$, we can substitute in (4) all the (v_i,θ_i) by their mean (v,θ_v), where $\theta_v = \sum_i \theta_i$. Finally, the problem is to find 3 numbers θ_B, θ_v, v such that they minimize r under the constraints :

(1) $\theta_B, \theta_v \geq 0$, $\theta_B+\theta_v = 1$, $-\frac{1}{2} \leq v \leq 0$

(4) $\frac{1}{2}\theta_B - 2\frac{n}{m}v^2\theta_v \geq \frac{1}{2} - r$

(6) $-\frac{1}{2}\theta_B - \frac{n}{m}v\theta_v \geq 0$

8.6 The solution in the case $m < n$

We must find $\theta_A, \theta_B, \theta_C, \theta_u$ which minimize r under the constraints :

(1) $\theta_B, \theta_C, \theta_u \geq 0$, $\theta_B + \theta_C + \theta_u = 1$, $0 \leq u \leq \frac{1}{2}$

(4) $\frac{1}{2}(\theta_B - \theta_C) - 2\frac{m}{n}u^2\theta_u \geq \frac{1}{2} - r$

(6) $\frac{1}{2}(-\theta_B + \theta_c) + \theta_u u \geq 0$

These constraints may be equivalently written:

(4) $r \geq \frac{1}{2} - \frac{1}{2}(\theta_B - \theta_C) + 2\frac{m}{n}u^2\theta_u$

(6) $\theta_u u \geq \frac{1}{2}(\theta_B - \theta_c)$

The solution requires that (4) is an equality. If $\theta_B = \theta_C = \frac{1}{2}$ and $\theta_u = 0$, we find that $r = \frac{1}{2}$. The minimal value of r is smaller or equal to $\frac{1}{2}$. Hence the optimal value of $(\theta_B - \theta_C)$ is positive or equal to 0. Since r is an increasing function of u, it appears that the minimal value of r is obtained when (6) is an equality.

Define $p = \frac{m}{n}$, and $x = \theta_B - \theta_C$. We must find $x, u, \theta = \theta_u$ which minimize r, where

(i) $r = \frac{1}{2} - \frac{1}{2}x + 2pu^2\theta$

(ii) $u = \frac{x}{2\theta}$

(iii) $0 \leq u \leq \frac{1}{2}$, $0 \leq \theta \leq 1$, $0 \leq x \leq (1-\theta)$

We can substitute $\frac{x}{2\theta}$ to u in (i) if $0 \leq \frac{x}{2\theta} \leq \frac{1}{2}$, that is if $x \leq \theta$. The problem then reduces to minimizing r under the constraints :

$r = \frac{1}{2}p\frac{x^2}{\theta} - \frac{1}{2}x + \frac{1}{2}$

$0 \leq \theta \leq 1$, $0 \leq x \leq Min(\theta, 1-\theta)$

If θ is fixed, the non-constrained minimum of r is obtained when $x = \widehat{x} = \frac{\theta}{2p}$. But this value does not always coincide with the optimal value x^*.

Several cases are to be investigated:

Case 1: (a)$\widehat{x} \leq \theta$ and (b)$\widehat{x} \leq (1-\theta)$

That is (a) $p \geq \frac{1}{2}$ and (b) $\theta \leq \frac{2p}{2p+1}$. In this case, $x^* = \widehat{x}$, and $r = \frac{1}{2} - \frac{1}{8p}\theta$. The value of r is decreasing with θ and reaches its minimum when $\theta = \frac{2p}{2p+1}$.

Case 2: (a)$\widehat{x} \leq \theta$ and (b')$\widehat{x} \geq (1-\theta)$

That is (a) $p \geq \frac{1}{2}$ and (b') $\frac{2p}{2p+1} \leq \theta \leq 1$. In this case, $x^* = (1-\theta)$ and $r = \frac{1}{2}\theta - p + \frac{1}{2}p\theta + \frac{1}{2}\frac{p}{\theta}$ and $\frac{\partial r}{\partial \theta} = \frac{(p+1)^2}{2\theta^2}\left(\theta^2 - \frac{p}{p+1}\right)$.

The value of r is minimized when $\theta = \sqrt{\frac{p}{p+1}}$, and we have $\frac{2p}{2p+1} \leq \sqrt{\frac{p}{p+1}} \leq 1$.

It appears from both cases 1 and 2 that, if $p \geq \frac{1}{2}$ then the optimal value are :

$$\begin{cases} \theta = \sqrt{\frac{p}{p+1}} \\ x = 1 - \theta \\ r = \sqrt{p(p+1)} - p \end{cases}$$

Case 3: (a') $\widehat{x} \geq \theta$ and (b) $\widehat{x} \leq (1-\theta)$

That is (a') $p \leq \frac{1}{2}$ and (b) $\theta \leq \frac{2p}{2p+1}$. In this case, $x^* = \theta$ and $r = \frac{1}{2} - \frac{1}{2}\theta(1-p)$. The value of r is decreasing with θ and reaches its minimum when $\theta = \frac{2p}{2p+1}$.

Case 4: (a')$\widehat{x} \geq \theta$ and (b')$\widehat{x} \geq (1-\theta)$

That is (a') $p \leq \frac{1}{2}$ and (b') $\frac{2p}{2p+1} \leq \theta \leq 1$. Then $\widehat{x}$ is not the optimal value of x. We distinguish two sub-cases.

- if $\frac{2p}{2p+1} \leq \theta \leq \frac{1}{2}$, the optimal value of x is $x^* = \theta$. This situation is similar to case (3): the optimal value of θ is $\theta = \frac{1}{2}$.

- if $\frac{1}{2} \leq \theta \leq 1$, the optimal value of x is $x^* = 1 - \theta$. This case is similar to case (2): the unconstrained optimal value of θ is $\theta = \sqrt{\frac{p}{p+1}}$. The optimal value of θ is thus $\theta = \sqrt{\frac{p}{p+1}}$ if $\sqrt{\frac{p}{p+1}} \geq \frac{1}{2}$, and $\theta = \frac{1}{2}$ if $\sqrt{\frac{p}{p+1}} \leq \frac{1}{2}$. That is, the optimal value of θ is $\theta = \sqrt{\frac{p}{p+1}}$ if $\frac{1}{3} \leq p \leq \frac{1}{2}$, and $\theta = \frac{1}{2}$ if $0 \leq p \leq \frac{1}{3}$.

To summarize the case where $m < n$, we find that, with $\alpha = \frac{m}{m+n}$:

if $\frac{1}{4} \le \alpha \le \frac{1}{2}$, then $\begin{cases} \theta = \sqrt{\frac{\frac{m}{n}}{\frac{m}{n}+1}} \\ r = \sqrt{\frac{m}{n}(\frac{m}{n}+1)} - \frac{m}{n} \\ r = \frac{\sqrt{\alpha}-\alpha}{1-\alpha} \end{cases}$

if $\alpha \le \frac{1}{4}$, then $\begin{cases} \theta = \frac{1}{2} \\ r = \frac{1}{4}\frac{m}{n} + \frac{1}{4} \\ r = \frac{1}{4(1-\alpha)} \end{cases}$

8.7 The solution in the case $m > n$

Defining $q = \frac{n}{m}$, one looks for $\theta = \theta_v \in [0,1]$ and $0 \le v \le \frac{1}{2}$ which minimize r under the constraints :
(4) $\frac{1}{2}(1-\theta) - 2qv^2\theta \ge \frac{1}{2} - r$
(6) $-\frac{1}{2}(1-\theta) + qv\theta \ge 0$
These constraints can be rewritten :
(4) $r \ge \frac{1}{2}\theta + 2q\theta v^2$
(6)$v \ge \frac{1}{2}\frac{1}{q}\left(\frac{1-\theta}{\theta}\right)$
The solution is such that (4) is an equality: r is thus an increasing function of v which is minimized when (6) is an equality.

From (6), we deduce that θ must be such that $\frac{1}{2}\frac{1}{q}\left(\frac{1-\theta}{\theta}\right) = v \le \frac{1}{2}$. We then look for a minimum of r when $r = \frac{1}{2\theta q}\left(\theta^2 - 2\theta + q\theta^2 + 1\right)$, $\frac{1}{1+q} \le \theta \le 1$.

By computing the derivative $r'_\theta = \frac{1}{2\theta^2 q}\left(\theta^2 + q\theta^2 - 1\right)$, one gets that r admits a global minimum when $\theta = \sqrt{\frac{1}{q+1}} \ge \frac{1}{1+q}$. This value fulfills the constraints, so we get:

$$\begin{cases} \theta = \sqrt{\frac{m}{n+m}} = \sqrt{\alpha} \\ r = \frac{1}{\frac{n}{m}}(\sqrt{1+\frac{n}{m}} - 1) \\ r = \frac{\alpha}{1-\alpha}\left(\sqrt{\frac{1}{\alpha}} - 1\right) \end{cases}$$

This completes the proof of Theorem 1.

8.8 Appendix C : proof of theorem 2

The proof essentially follows the strategy used in order to prove Theorem 1, namely the use of composed block matrices. The main difference is that we no longer distinguish between the 2 sets M and N.

Let X be a vote profile involving J individuals (columns), C issues (lines), and such that, without loss of generality, $\forall c$, $\mu^c(X) = 0$.

We denote $\begin{cases} s(c) = \frac{m_c^0 - m_c^1}{J}, \text{ where } c = 1, ..., C \\ J(\gamma, X) = \text{proportion of issues } c \text{ such that } s(c) \le \gamma \end{cases}$

Moreover, we assume that $(\mathcal{P})\begin{cases} J(\gamma, X) = x \\ \forall j \ne h,\ d(x_j, x_h) \le rC \end{cases}$

First note that, for any permutation σ of the voters, $(\mathcal{P})$ also holds for the permuted profile X_σ, as well as for the composition of all X_σ, where $\sigma \in \sum$, the set of permutations of $\{1, ..., J\}$. Hence, we can assume w.l.g. that X is a composition of block matrices, where Y is the initial matrix containing C' rows. Pick up line c in Y, and suppose that $m_c^1(Y) = k$. Moreover, define $u_k = \frac{1}{2} - \frac{k}{J}$. Furthermore, let $c(k)$ be the number of rows c' in Y such that $m_{c'}^1(Y) = k$, and let $\theta_k = \frac{c(k)}{C'}$.

Since $\mu^c(X) = 0$ for all c, then $u_k \ge 0$.

The distance between two ballots x_j and x_h in X is given by $r'(J!)$, where $r' = (\frac{1}{2} - 2\sum_k u_k^2\theta_k)\frac{J}{J-1}$.

We then have to solve the following program:

$Min\ r$ under the constraints

$$\begin{cases} r = (\frac{1}{2} - 2\sum_k u_k^2\theta_k)(\frac{J}{J-1}) \\ \sum_{k:2u_k\leq\gamma} \theta_k = x \end{cases}$$

One get $r = \frac{J}{J-1}[\frac{1}{2} - 2\sum_{k:2u_k\leq\gamma} u_k^2\theta_k - 2\sum_{k:2u_k>\gamma} u_k^2\theta_k]$. In order to minimize r requires maximizing $\sum_k u_k^2\theta_k$ and then setting $\begin{cases} u_k = \gamma/2 \text{ if } 2u_k \leq \gamma \\ u_k = \frac{1}{2} \text{ if } 2u_k > \gamma \end{cases}$.

One thus obtains:

- $r = \frac{J}{J-1}(\frac{1}{2} - 2\frac{\gamma^2}{4}x - 2\frac{1}{4}(1-x)) = \frac{J}{J-1}\frac{x(1-\gamma^2)}{2}$
- $x = 2\frac{J-1}{J}\frac{r}{1-\gamma^2}$

The conclusion follows from $Lim_{m\to\infty}[2\frac{J-1}{J}\frac{r}{1-\gamma^2}] = \frac{2r}{1-\gamma^2}$.